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On YM_2 measures and area-preserving diffeomorphisms

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Abstract

For a given gauge group and compact Riemannian two-manifold, it is known that the associated Yang–Mills measure can be defined directly as a finitely additive measure on the space of connections, and this finitely additive measure is invariant with respect to $SDiff$, the group of all area-preserving diffeomorphisms of the surface. The first question we address is whether this symmetry essentially characterizes the projection of the Yang–Mills measure to the space of gauge equivalence classes. The proper formulation of this question entails the construction of an $SDiff$-equivariant completion of the space of continuous connections, such that the projection of the Yang–Mills measure to the space of gauge equivalence classes has a countably additive extension. We also consider the coupling of the Yang–Mills measure to determinants of Dirac operators. The basic problems are to prove that the coupled measure is absolutely continuous with respect to the background Yang–Mills measure, to find a reasonable formula for the Radon–Nikodym derivative, and to analyze the action of $SDiff$.

Keywords: Yang–Mills measures; Wiener measures; Zeta determinants; $SDiff$
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0. Introduction

0.1. The YM_2-measure

To discuss the YM_2-measure we need the following data:

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$$\begin{aligned} K &= \textit{connected compact Lie group}, \\ \langle\ ,\ \rangle &= \textit{Ad}\,K\textit{-invariant inner product}, \\ \Sigma &= \textit{compact 2-manifold (possibly with boundary)}, \\ \omega &= \textit{finite area form on}\,\Sigma, \\ P \to \Sigma &= \textit{principal K-bundle}. \end{aligned} \tag{0.1}$$

The Yang–Mills measure associated to the data in (0.1) is written heuristically as

$$\mathrm{d}\nu^{\omega}_{YM_2}(A) = \exp\left(-\frac{1}{2}\int_{\Sigma}\langle F_A \wedge *F_A\rangle\right)\mathcal{D}A, \tag{0.2}$$

where A is a connection on P, $\mathcal{D}A$ denotes the formal Lebesgue measure on the space of connections of P, and $*F_A = F_A/\omega$. In this two-dimensional setting it appears that this measure, applied to gauge invariant functions, has an unambiguous meaning (see, e.g. [GKS,Dr,KlK,Sen]).

The starting point of this paper will be a known direct definition of $\nu_{YM} = \nu^{\omega}_{YM_2}$ as a finite, finitely additive measure, which we will recall in Section 2. The existence of this direct definition is usually expressed by saying that the Villain lattice approximation to YM_2 is exact; this is undoubtedly the most remarkable aspect of two-dimensional Yang–Mills theory. However the measure ν_{YM} cannot be extended to a countably additive measure in a natural way. For this reason, in addition to the usual physical considerations, it is the projection $\pi_*\nu_{YM}$ of ν_{YM} to the space of gauge equivalence classes of connections which is the primary object of interest in this paper.

Our first objective is to extend $\pi_*\nu_{YM}$ to a countably additive measure and describe its support in geometric terms. This is nontrivial because there does not appear to be a natural way to view $\pi_*\nu_{YM}$ as a countably additive measure on a space of equivalence classes of differential geometric connections, even in the distributional sense ("natural" essentially means equivariant with respect to automorphisms). The approach which we will advocate is to map the space of connections to, what we will choose to call, the space of gluon potentials. The idea is that a continuous connection, in the sense of differential geometry, is determined by the corresponding functor of parallel translation along all C^1-curves of the base space. The space of ($C^{-\alpha}$) gluon potentials is simply the space of all functors (dual to C^{α}-paths) with the formal properties of parallel translation.

In the language of Physics, this correspondence amounts to viewing gauge fields as constrained chiral fields on a loop space (see Ch. 7 of [Pol]). In the language of Probability, this amounts to viewing $\pi_*\nu_{YM_2}$ as a certain continuous version of a stochastic process (holonomy) indexed by closed loops. In Geometry this idea has been used by Driver to parameterize isomorphism classes of principal bundles with connections (see [Dr2]), and by Ashtekar and collaborators in the investigations of quantum gravity and knot invariants (see [AL]). Our claim, which we have not completely substantiated, is simply that this point of view is technically useful in understanding YM_2 (this is also suggested in [KlK]; the question of whether this (or some related) point of view is useful for YM_3 or YM_4 is a fundamental unresolved issue; see [Gr,Dr] for further discussion).

Once this point of view is adopted, ideas of Dudley can be used to give a very plausible argument that $\pi_* \nu_{YM}$ is supported on the space of $C^{-\alpha}$-gluons (gauge equivalence classes of gluon potentials dual to C^{α}-curves), for any $\alpha > 1$; the argument we give is completely rigorous only in the abelian case. The intuition behind this is the following. In the abelian case the formal measure (0.2), in some gauge, can be viewed as a Gaussian measure corresponding to the W^1-norm of the connection A. In our two-dimensional setting, this means that YM_2 (in our fixed gauge) has measure zero on pointwise defined connections, but full measure on distributional $W^{-\epsilon}$-connections, for any $\epsilon > 0$. Since continuous connections define C^{-1}-gluons, it is not surprising that the YM_2-measure is supported on $C^{-\alpha}$-gluons, for any $\alpha > 1$.

By choosing coordinates for the surface appropriately, one can express $\pi_* \nu_{YM}$ as a certain (finite codimensional) conditioned measure associated to an iterate of Wiener measure on the double path space of the gauge group. The disintegration of $\pi_* \nu_{YM}$ relative to the natural fibration of this space is of intrinsic interest, for it provokes speculation about extensions to higher dimensions. This development is hinted at by the formal expressions in [Fine].

The main original objective of this work was to find a characterization of $d = 2$ Yang–Mills measures in terms of their symmetry. At this point we can only offer a conjecture. It is always the case that $\pi_* \nu_{YM}^{\hbar\omega}$, for each value of Planck's constant, is invariant with respect to the natural action of $SDiff(\Sigma)$ on gluons (when $\hbar = 0$, $\pi_* \nu_{YM}^{\hbar\omega}$ should be interpreted as the canonical symplectic volume element on the moduli space of classical solutions). We conjecture that, modulo the possibility of reducing the structure group, the $\pi_* \nu_{YM}^{\hbar\omega}$, $\hbar \geq 0$ are (up to a multiple) the only finite ergodic $SDiff(\Sigma)$-invariant measures on the space of gluons. This would be an attractive form of the assertion that the YM_2-measure is unambiguously defined. When $\hbar = 0$ this reduces to the assertion that the ergodic invariant probabilities for the action of the mapping class group on the space of classical solutions are parameterized by the structure group; this assertion completely determines the form of the classical limit, which was computed directly in [Fo].

0.2. *Line bundles*

The basic observables of YM theory are Wilson loops, and in this paper we have codified this by viewing $\pi_* \nu_{YM}$ as a measure on the space of gluons. However as soon as one attempts to couple YM to fermions, or to use YM_2 as a stepping stone to understanding YM_3 + *Chern–Simons*, then one encounters more sophisticated random variables. We will first explain how these arise from a purely mathematical point of view; in Section 0.3 we will briefly reconsider this from the point of view of Physics.

Suppose that Σ is closed and oriented. Let $\mathcal{A}$ denote the affine space of K-connections on P, and let $\mathcal{C}$ denote the space of gauge equivalence classes. Subtleties aside, the space $\mathcal{C}$ is the classifying space for the gauge group $\mathcal{K}$ of P. Line bundles are classified essentially by $\pi_2 K$.

For simplicity of exposition, suppose that K is simply connected and has a simple Lie algebra. In this case line bundles are parameterized by $\mathbb{Z}$, and a generator can be constructed using the Wess–Zumino–Novikov–Witten cocycle, following Mickelsson [Mi], or equivalently, the Chern–Simons functional, as in [RSW]. This bundle comes equipped with a hermitian structure (because the cocycle is unitary). In a completely explicit manner, a power of this generator can be identified with the determinant line bundle of $\bar{\partial}$-operators coupled to connections on AdP, relative to a choice of Riemannian spin structure on the surface. In this realization the hermitian structure is realized by the Quillen metric, which involves Ray–Singer regularization of determinants of Laplace-type operators. Combining this hermitian structure with the YM_2-measure, we should formally have unitary representations

$$SDiff(\Sigma) \times L^2(Det\,\bar{\partial}^{\otimes s} \to C). \tag{0.3}$$

The analytic obstruction is that $\pi_* \nu_{YM}$ is not supported on the space of connections for which the hermitian structure is defined. To be quite precise, the Mickelsson cocycle shows that the hermitian structure is defined on gauge equivalence classes of continuous connections (essentially C^{-1}-gluons), while the measure is defined on $C^{-\alpha}$-gluons, for $\alpha > 1$. The upshot is that there is a logarithmic divergence.

There are at least two conceivable ways of proceeding to define the representation (0.3). One is to regularize (in an equivariant way) the divergent expression for the integrand in the integral defining the inner product, e.g.

$$|det\,\bar{\partial}_a|^{2s}\, \mathrm{d}\pi_*\nu_{YM}([A]) = \exp(-s\zeta_a'(0))\, \mathrm{d}\pi_*\nu_{YM}([A]), \tag{0.4}$$

where $det\,\bar{\partial}_a$ is the canonical section, and a is the (0,1) part of A. Our objective is to show that the total expression

$$(1/E)|det\,\bar{\partial}_a|^{2s}\, \mathrm{d}\pi_*\nu_{YM}([A]) \tag{0.5}$$

can be properly defined as a measure; in fact, because the divergence is mild, we strongly suspect that this measure is absolutely continuous with respect to $\pi_*\nu_{YM}$ (this is certainly true in the abelian case). Furthermore this measure, viewed more abstractly as a measure with values in the bundle $|Det\,\bar{\partial}|^{-2s}$, should be invariant with respect to $SDiff(\Sigma)$, so that we do obtain a unitary representation as in (0.3). As in the case of the support problem, our arguments are only very plausible.

The main step in making sense of (0.5) is to find an analytically tractible expression for the ζ-determinant. In the case of S^2, we can generically write

$$A = g^{*-1}(\partial g^*) - (\bar{\partial} g)g^{-1}, \tag{0.6}$$

where g is a map into G, the complexification of K. The gauge equivalence class of A then corresponds to $\Omega = g^*g$. In terms of this coordinate for the moduli space of connections, the measure (0.5) is heuristically written as

$$\frac{1}{E}\exp(\mathcal{E}(\Omega) + \mathrm{i}WZW(\Omega))\exp\left(-\frac{1}{2}\int\langle\bar{\partial}(\Omega^{-1}\partial\Omega)\wedge *\bar{\partial}(\Omega^{-1}\partial\Omega)\rangle\right)\mathcal{D}\Omega, \tag{0.7}$$

where the first exponential factor is the expression for the ζ-determinant (involving the energy and the Wess–Zumino functionals), and the second is the density for the Yang–Mills measure.

A possible alternate method would be to directly evaluate the spherical function

$$SDiff(\Sigma) \to \mathbb{C}: \ \phi \to \langle \phi_* det\ \bar{\partial}, det\ \bar{\partial} \rangle; \qquad (0.8)$$

this would be of considerable interest in Physics, as we will explain below.

0.3. *Interfaces with Physics*

The quantum field theory corresponding to YM_2 is in many ways quite simple; in particular the space of states is the Hilbert space of central functions on the gauge group, and the time evolution, corresponding to an annulus of area A, is given by the heat semigroup at time A. However when one considers the theory on more general surfaces, while it remains transparent at the level of the Villain approximation (so that the partition function can be calculated), it is nontrivial at the level of holonomy, principally because there is a complicated classical limit (see [Wi1] for an account of this).

Coupling YM_2 with (possibly massive) fermions basically amounts to making sense of the path integral measure

$$\left(\int \exp(-\bar{\psi} D_{A,m} \psi) \mathcal{D}\psi\right) \exp\left(-\frac{1}{2}\int \langle F_A \wedge *F_A\rangle\right) \mathcal{D}A$$
$$= det(D_{A,m}) \exp\left(-\frac{1}{2}\int \langle F_A \wedge *F_A\rangle\right) \mathcal{D}A, \qquad (0.9)$$

where $D_{A,m}$ is the full Dirac operator (possibly with mass) coupled to the gauge potential (one cannot straightforwardly couple YM_2 to the chiral Dirac operator, because of the existence of an anomaly; see [Mi2]).

In the massless case, it seems reasonable to believe that this theory can be explicitly constructed and computed. In particular the space of states (corresponding to a circle) is known (see [Mi2]), and it appears that the Hamiltonian has been rigorously constructed (see [Mi2,LS]). From the path integral point of view, one begins with the formal identity

$$det(D_A) = det|\bar{\partial}_a|^2;$$

one is tempted to interpret the latter determinant as the Ray–Singer ζ-regularized determinant, so that the partition function is equal to the integral of a regularized version of (0.4). The computability of the theory is, in this interpretation, closely related to the representation-theoretic problem of computing the spherical function (0.8).

Of course the ultimate justification of this (assuming that it all fits together) will have to emerge from an analysis of lattice gauge theory approximations, since this is the basis of intuition about the physical content of gauge theories. In the massive case, [KlK] has shown the existence of a path integral, by considering a limit of lattice approximations. However at present there does not seem to be any real insight into the geometric meaning of this limit.

A second interface of this theory with Physics involves $YM_3 + k * Chern\text{–}Simons$. The space of states for this theory, corresponding to a closed spin 2-manifold Σ, is thought to be the quantization of the space T^*C, where the symplectic structure is the sum of the canonical structure plus k-times the pullback from C of a specific form that represents a generator for $H^2(C, \mathbb{Z})$ (see [AM]). Thus it should be possible to realize this state space as the space of sections of the line bundle $(Det\ \bar{\partial})^{\otimes s}$, equipped with an appropriate unitary structure. It would be remarkable if this unitary structure turned out to be identical to the one we conjecture exists by virtue of (0.3), since this has more symmetry than one would naively expect.

0.4. Organization of the paper

The paper is currently organized as follows: In Section 1 we discuss the definitions of a gluon potential and gluon, and what little we can say about the topology of the space of gluons. One might expect that the space of gluon potentials is contractible, and that the space of gluons is homotopy equivalent to the space of gauge equivalence classes of connections, but for the present this is a mystery. An important general fact is that a based gluon can be viewed as a holonomy functor, i.e. essentially as a homomorphism from loops based at a given point into K. This is a direct extension of ideas in [Dr2]. We also consider gluon potentials which are only defined on generators coming from a coordinate system (this is a coordinate dependent completion of the space of gluons). This is essential in understanding the connection between $\pi_* \nu_{YM}$ and iterates of Wiener measure. In an appendix we have indicated how these coordinate dependent considerations extend directly for higher-dimensional spheres.

In Sections 2.1 and 2.2 we recall the definitions of the YM_2-measure and iterates of Wiener measure, respectively, and in Section 2.3 we discuss how they are related. In Section 2.6 we discuss the countable additivity problem.

In Section 3 we present some obvious conjectures about the decomposition of the natural L^2-representation of area-preserving diffeomorphisms, and the ergodicity of $\pi_* \nu_{YM}$. In Section 3.2 we formulate a conjecture to the effect that $\pi_* \nu_{YM}$ can be characterized by its invariance with respect to area-preserving diffeomorphisms.

In Section 4 we carefully identify the Mickelsson realization of (a power of) the canonical line bundle over the moduli space of connections with Quillen's realization of the bundle using $\bar{\partial}$ operators. In particular this yields relatively explicit formulas for certain ζ-function determinants. The material of this section is of intrinsic geometric interest, because it suggests a way of viewing the so-called nonabelian theta functions as bonafide functions on the space of connections.

In Section 5 we discuss the existence of the measures which are given formally by (0.5) (or (0.9)), and their relation to other natural measures.

0.5. Notational conventions

We will need to consider several different path spaces. Given a space X, $Path^{p,q} X$ will denote the space of paths from p to q in X; it will be understood that the paths are C^0, i.e. continuous, unless we indicate some other degree of smoothness by a subscript, and the domain of a path is I, the unit interval. Similarly $Path^{*,*}X$ will denote the space of all paths in X, and so on.

For technical reasons, we will also need to consider a subcategory of C^{α}-paths, denoted $ResPath_{C^{\alpha}}$. By definition $c \in ResPath_{C^{\alpha}}$, i.e. c is a restricted path, if c is a C^{α}-embedding of $I \setminus S$, where S consists of finitely many points. We only need this in the proof of Conjecture 2.10, and it is an interesting question whether it can be dispensed with.

1. The space of gluons

Let $\mathcal{A}_C$ denote the space of continuous K-connections on P, in the sense of differential geometry, and let $\mathcal{C}_C$ denote the space of gauge equivalence classes of continuous K-connections, relative to the group of C^1-gauge transformations, $\mathcal{K}_{C^1}$. There are two completions of these spaces which we will need; they can be represented schematically by the following diagram:

$$\begin{array}{ccccc} \mathcal{K}_{W^1} & \leftarrow & \mathcal{K}_{C^1} & \rightarrow & \mathcal{K}_{C^0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_{W^0} & \leftarrow & \mathcal{A}_C & \rightarrow & \mathcal{P}otentials_{C^{-1}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_{W^0} & \leftarrow & \mathcal{C}_C & \rightarrow & \mathcal{G}luons_{C^{-1}} \end{array} \tag{1.1}$$

The spaces in the left column arise because they are the maximal domains of several natural differential geometric constructions; for example, if the surface Σ is oriented, then the space $\mathcal{A}_{W^0}$ has a natural symplectic structure given by

$$\underline{\omega}_\theta(\eta_1, \eta_2) = \int_\Sigma \langle \eta_1 \wedge \eta_2 \rangle, \tag{1.2}$$

where $\eta_1, \eta_2 \in \Omega^1_{W^0}(\Sigma, AdP) \cong T\mathcal{A}_{W^0}|_\theta$. (But note that our favored Hamiltonian, the Yang–Mills functional (which requires the additional structure of a volume element), is only defined on the space of W^1-connections, as is the moment map for the action of $Aut(P)$ on $\mathcal{A}$,

$$\mu : \mathcal{A} \rightarrow (Aut P)^*: \ \theta \rightarrow \Omega_\theta, \tag{1.3}$$

where curvature Ω_θ is interpreted as the linear functional

$$\Omega_\theta(v) = \int_\Sigma \langle \Omega_\theta \wedge \theta(v) \rangle.) \tag{1.4}$$

The purpose of this section is to define the spaces on the right, and related spaces, and to describe their topology. In Appendix A, we have indicated how our considerations extend directly to S^3 and S^4.

1.1. The definition of a gluon

Let $Paths_{C^k}(\Sigma)$ denote the involutive topological category for which the objects are the points of Σ, the morphisms are the C^k-paths between points, and the involution is reversing the parameterization. There is a natural action of path reparameterization by the group $Homeo^+_{C^k}(I)$.

Let $Mor(P)$ denote the involutive topological category for which the objects are the fibers of P, viewed as K-homogeneous spaces, the morphisms are the K-equivariant maps between fibers, and the involution is inversion.

Definition 1.1. A C^{-k} P-gluon potential is a continuous reparameterization invariant functor from $Path_{C^k}(\Sigma)$ to $Mor(P)$. The set of all C^{-k} P-gluon potentials will be denoted by $\mathcal{P}otentials_{C^{-k}}$, and the space of $\mathcal{K}_{C^0}$-gauge equivalence classes will be denoted by $\mathcal{G}luons_{C^{-k}}$ (to indicate the dependence upon P, if necessary, we will write $\mathcal{G}luons(P)$, etc.).

Remark 1.2.

(1) The meaning of continuity in this context is the following: If x_j is a sequence of paths converging to a path $x : I \to \Sigma$ in C^k, then $g_{x_j} \in Mor(P_{x_j(0)}, P_{x_j(1)})$ converges to $g_x \in Mor(P_{x(0)}, P_{x(1)})$. The action

$$\mathcal{K}_{C^0} \times \mathcal{P}otentials_{C^{-k}} \to \mathcal{P}otentials_{C^{-k}} \tag{1.5}$$

is given by

$$k, g \to k_* g, \quad \text{where } (k_* g)_x = k(x(1)) g_x k(x(0))^{-1}. \tag{1.6}$$

(2) The domain of definition of a C^{-k}-gluon potential can always be extended to piecewise C^k-paths in a canonical way. We will use this extension often, and without comment.

(3) A continuous connection A on P defines a C^{-1} P-gluon potential $g = g^A$, where g maps a path to parallel translation from the initial fiber to the final fiber. A gluon potential coming from a continuous connection has two special properties:

(a) *Differentiability*: Parallel translation along a path is C^1; that is, if $x(t)$ is a C^1-curve in Σ, then for $p \in P_{x(0)}$ the lift

$$p^x : t \to g_{(s \to x(st))}(p) \tag{1.7}$$

is C^1 in P.

(b) *Locality*: If $x(t)$ and $y(t)$ are equal to first order at $x(0)$, then for $p \in P_{x(0)}$ the lifts p^x and p^y are equal to first order.

These conditions guarantee the existence of the cross-section

$$c_p : T\Sigma|_{\pi(p)} \to TP|_p\colon\ x'(0) \to \frac{\mathrm{d}}{\mathrm{d}t} p^x(t)\Big|_{t=0}, \tag{1.8}$$

which determines the connection. I believe that conditions (a) and (b) characterize the image of $\mathcal{A}_C$ in $\mathcal{P}otentials_{C^{-1}}$.

(4) Note that

$$\mathcal{A}_C \subset \mathcal{P}otentials_{C^{-1}} \subset \cdots \subset \mathcal{P}otentials_{C^{-k}} \subset \cdots \subset \mathcal{P}otentials_{C^{-\infty}}, \quad (1.9)$$

so that gluon potentials are analogous to distributional connections, but with the important difference that parallel translation is defined for gluon potentials. Note in particular that parallel translation is not defined for connections in the class W^0.

(5) In the definition above, we could consider the category of restricted paths instead of all paths. We will refer to these generalized objects as gluon potentials dual to restricted paths.

A gluon potential is a map from paths to morphisms of P that satisfies a very large number of constraints. If we could find a primitive system of generators for the category $Path^{*,*}_{C^k}(\Sigma)$, then we could more simply regard a gluon potential as a function on this subset with values in $Mor(P)$, satisfying a more manageable set of constraints. This does not appear to be feasible, but we will nonetheless find it useful to consider generalized gluon potentials that are defined only on a subset of all regular paths. In particular, in Section 1.3 we will study a coordinate dependent completion of the space of gluons, which we will refer to as a space of coordinate based gluons. We will give the definition here.

Suppose that

$$z : U \to D \quad (1.10)$$

is a coordinate for Σ, where $\Sigma \backslash U$ is a piecewise smooth 1-simplex. The coordinate segments

$$t \to tre^{i\theta} \quad \text{and} \quad t \to re^{it\theta} \quad (1.11)$$

generate a dense subset of $Path^{*,*}_{PC^k}(\Sigma)$, for any k. Hence a gluon potential g defines a pair of functions, by restriction,

$$g^r(r,\theta) = g_{\{t \to tre^{i\theta}\}}, \quad \text{and} \quad g^\theta(r,\theta) = g_{\{t \to re^{it\theta}\}}, \quad (1.12)$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$; g is completely determined by its restriction to these curves.

Definition 1.3. A coordinate based gluon potential is a pair of continuous functions g^r and g^θ satisfying the constraints that apply to the restriction of a gluon potential. The set of all P-gluon potentials will be denoted by $Potentials^{(coord)}$, and the space of $\mathcal{K}_{C^0}$-gauge equivalence classes will be denoted by $\mathcal{G}luons^{(coord)}$.

The advantage of this completion is that it can be described very explicitly, as we will see in Section 1.3; its disadvantage is that it is not equivariant.

1.2. Gluons as holonomy functors, and topology

Fix a base point $x_0 \in \Sigma$, and temporarily forget about the principal bundle $P \to \Sigma$.

Definition 1.4. A based C^{-k}-gluon is a reparameterization invariant functor

$$h : Path^{x_0,x_0}_{PC^k}(\Sigma) \to Mor(K).$$

For each topological type of bundle $[P] \in [\Sigma, BK]$, if we fix a representative $P \to \Sigma$ and an identification of P_{x_0} and K, then there is a map given by holonomy at x_0,

$$holonomy : \bigsqcup_{[P]\in[\Sigma,BK]} \mathcal{G}luons_{based,C^{-k}}(P) \to \{basedC^{-k}\text{-}gluons\}. \tag{1.13}$$

The following is a direct extension of Driver's parameterization of isomorphism classes of bundles with connections in [Dr2].

Proposition 1.5. *The map* (1.13) *is a bijection.*

Proof. Suppose that g_1 and g_2 are gluon potentials for the same bundle P, and suppose that $h_1 = h_2$, where h_j is the holonomy of g_j at x_0. Then the equation

$$k(c_1)(g_1)_c = (g_2)_c \tag{1.14}$$

implicitly defines a based gauge transformation k with $k_* g_1 = g_2$. This proves that (1.13) is 1–1 on each component.

For each $x \in \Sigma$ we can find an open subset U such that $x_0, x \in U$ and there is a smooth contraction of U to x_0. Choose a covering of Σ by such open sets U^a, and for each a, let

$$c^a : I \times U^a \to U^a : (t, x) \to c^a_t(x) \tag{1.15}$$

denote a contraction with $c^a_1(x) = x$ and $c^a_0(x) = x_0$, for all $x \in U^a$.

Now suppose that h is a based C^{-k}-gluon. Define transition functions for a C^0 K-bundle in the following way:

$$k_{ab} : U^a \cap U^b \to K : x \to h_{\{t\to c^a_t(x)^{-1}\cdot c^b_t(x)\}}. \tag{1.16}$$

If h comes from a P-gluon, this clearly shows that we recover $[P]$, completing the proof that (1.13) is 1–1. In the general case let P denote the bundle defined by these transition functions, and for each a let

$$\psi^a : P|_{U^a} \to U^a \times K \tag{1.17}$$

denote an isomorphism of K-bundles such that $k_{ab}(x)\psi^b|_x = \psi^a|_x$.

We will now define a C^{-k} P-gluon potential g such that holonomy maps g to h. Note that g will be completely determined once we specify how g maps P_{x_0} to P_x along each curve $t \to c^a_t(x)$, for each x. In terms of the coordinate (1.17) $g_{\{t\to c^a_t(x)\}}$ is the identity; i.e.

$$g_{\{t\to c^a_t(x)\}} = (\psi^a|_x)^{-1}(\psi^a|_{x_0}). \tag{1.18}$$

It follows immediately from this that

$$g^{-1}_{\{t\to c^a_t(x)\}} \circ g_{\{t\to c^b_t(x)\}} = h_{c^a_{(\cdot)}(x)^{-1}c^b_{(\cdot)}(x)}. \tag{1.19}$$

This consistency relation implies that g extends canonically to a gluon potential with holonomy h. This shows that (1.13) is surjective, completing the proof. □

We now turn to the structure of the spaces of gluons and gluon potentials.

There is a natural complete separable metric on the space $Potentials_{C^{-k}}$; it is given by

$$\rho(g_1, g_2) = \sup d((g_1)_c, (g_2)_c) \tag{1.20}$$

where the sup is over embedded C^k-paths c, and $d = d_{x,y}$ is the metric on $Mor(P_x, P_y)$ induced by the bi-invariant Riemannian structure corresponding to $\langle\cdot,\cdot\rangle$. The automorphisms of P which induce C^k-diffeomorphisms of the base act isometrically on this space.

Fix a basepoint x_0 and let $\mathcal{K}_{based,C^0}$ denote the group of C^0-gauge transformations which are trivial at x_0. This group acts on potentials; let $\mathcal{G}luons_{based,C^{-k}}$ denote the quotient. The metric ρ induces a pseudo-metric on gauge equivalence classes, which we will denote by $\bar{\rho}$. On the other hand, we can define a second metric ρ_2 on based gluons by taking the supremum in (1.20) over closed paths at x_0.

Proposition 1.6.

(a) *$\mathcal{K}_{based,C^0}$ acts freely and isometrically on $Potentials_{C^{-k}}$.*

(b) $\rho_2 \le \bar{\rho} \le 3\rho_2$.

(c) *The space $\mathcal{G}luons_{based,C^{-k}}$ is a complete separable metric space, and C^k-diffeomor phisms of Σ act isometrically with respect to both $\bar{\rho}$ and ρ_2.*

Proof. (a) and (c) are obvious.

To prove (b) we will use the open cover $\{U^a\}$ and contractions that were introduced in the proof of (1.13). For each a let

$$k^a(x) = q_{c_t^a(x)} \circ g^{-1}_{c_t^a(x)}. \tag{1.21}$$

Then

$$d(k^a(x), k^b(x)) = d\left(g^{-1}_{c_t^a(x)} \circ g_{c_t^b(x)}, q^{-1}_{c_t^a(x)} \circ q_{c_t^b(x)}\right) \le \rho_2([g], [q]). \tag{1.22}$$

Therefore there exists $k \in \mathcal{K}_{based,C^0}$ such that for each a

$$d(k(x), k^a(x)) \le \rho_2([g], [q]). \tag{1.23}$$

It is then easy to see that

$$\rho(k_* g, q) \le 3\rho_2([g], [q]). \tag{1.24}$$

This implies (b).

If $\rho([g], [q]) = 0$, then there are based gauge transformations k_n such that $d(k_n(c_1)g_c, q_c) \to 0$ uniformly for all knots beginning at x_0. This implies that k_n converges uniformly to k defined implicitly by

$$k(c_1) = q_c \circ g_c^{-1}. \tag{1.25}$$

Then k is a C^0 based gauge transformation, and $k_* g = q$ on all knots beginning at x_0, hence on all knots. Hence ρ is a metric, and this implies (b). □

We would like to believe that $Potentials_{C^{-k}}$ is a contractible space, and that

$$\pi : Potentials_{C^{-k}} \to \mathcal{G}luons_{based,C^{-k}} \tag{1.26}$$

has local cross-sections. This would imply that $\mathcal{G}luons_{based}$ is a model for the classifying space of $\mathcal{K}_{based}$. This seems reasonable, because in Section 1.3, where we will specialize to 2 dimensions, we will see that there is a composition of maps

$$\mathcal{C}_{based,C^0} \to \mathcal{G}luons_{based,C^{-k}} \to \mathcal{G}luons_{based}^{(coord)}, \tag{1.27}$$

which is a homotopy equivalence.

1.3. Coordinate based gluons and iterated path spaces

In this section we will parameterize the space of coordinate based gluon potentials and gluons, relative to a coordinate

$$U \to D, \tag{1.28}$$

as in (1.10). Our goal is to use the functions

$$g^r(r,\theta) = g_{\{t \to tre^{i\theta}\}} \quad \text{and} \quad g^\theta(r,\theta) = g_{\{t \to re^{it\theta}\}}, \tag{1.29}$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, to parameterize the space of coordinate based gluon potentials in terms of iterated path spaces. This can also be used to parameterize coordinate based gluons in a topologically transparent way. However, there is a second and more useful parameterization for coordinate based gluons in terms of holonomy. This is essentially what appears in [Fine]. To fix the ideas we first consider:

The disk. Fix a trivialization of P. In this case there is a bijection

$$\begin{aligned} &\mathcal{P}otentials^{(coord)} \leftrightarrow D_0K \times Path_r^{1,*}(Path_\theta^{1,*}K), \\ &g \leftrightarrow (g^r, g^\theta). \end{aligned} \tag{1.30}$$

Since the based gauge group of the disk is D_0K, we have

$$\mathcal{G}luons_{based}^{(coord)} \cong D_0K \times_{D_0K} Path_r^{1,*}(Path_\theta^{1,*}K), \tag{1.31}$$

a bundle with contractible fiber over a point. In this setting the radial gauge amounts to the standard identification of the fiber over the identity with the total space in (1.31),

$$Path_r^{1,*}Path_\theta^{1,*}K \to D_0K \times_{D_0K} Path_r^{1,*}Path_\theta^{1,*}K\colon \gamma \to [1,\gamma]. \tag{1.32}$$

A second parameterization of gluons, using holonomy, is given by

$$\begin{aligned} &\mathcal{G}luons_{based}^{(coord)} \cong Path_r^{1,*}Path_\theta^{1,*}K, \\ &[g] \leftrightarrow \left(g_{\partial\left\{\begin{smallmatrix} 0 \le \phi \le 2\pi\theta \\ 0 \le \rho \le \sqrt{r} \end{smallmatrix}\right\}} \right), \end{aligned} \tag{1.33}$$

where the origin is taken as the basepoint in the computation of holonomy. We have parameterized holonomy by $\sqrt{r}$ for technical convenience (see Section 2.3).

The sphere. We first assume that P is a trivial bundle. We identify the sphere with the disk, where the boundary points are all identified with the point at infinity. In this case the correspondence $g \leftrightarrow (g^r, g^\theta)$ defines bijections

$$\mathcal{P}otentials^{(coord)} \leftrightarrow D_0K \times Path_r^{1,1}(Path_\theta^{1,*}K), \tag{1.34}$$

$$\mathcal{G}luons_{based}^{(coord)} \cong D_0K \times_{\Omega_0^2 K} Path_r^{1,1} Path_\theta^{1,*}K. \tag{1.35}$$

The second space is a principal fiber bundle with base $(LK)_0/K$ and contractible fiber, hence is homotopy equivalent to the identity component of the based loop space of K.

In the general case, aside from notational complications, the only real change is that the zero component of the based loop space is replaced by the component of the space

$$L\,Mor(P_0, P_\infty)/Mor(P_\infty),$$

which represents the topological type of P. In particular it is easy to see that the space of coordinate based gluon potentials is contractible.

In general we can alternately describe based gluons using holonomy. We will implicitly use the isomorphism

$$L\,Mor(P_0, P_\infty)/Mor(P_\infty) \cong \Omega\,Mor(P_0)\colon\ \gamma_\theta \to \gamma_\theta^{-1} \circ \gamma_0$$

in the following:

Proposition 1.7. *The space of coordinate based P-gluons is a principal bundle with contractible structure group and base space*

$$B = P\text{-}component \subset \Omega\,Mor(P_0). \tag{1.36}$$

More precisely, the map $[g] \to h$ *defines an isomorphism*

$$\begin{aligned}&\mathcal{G}luons_{based}^{(coord)}(P)\\ &\quad\cong \bigsqcup_{\gamma \in B} Path_r^{1,(\theta \to \gamma_\theta^{-1} \circ \gamma_0)}(Path_\theta^{1,*}Mor(P_0))\\ &\quad= \bigsqcup_{\gamma \in B} \{h(r,\theta) \in Path^{1,*}Path^{1,*}Mor(P_0)\colon\ h(1,\theta) = \gamma_\theta^{-1} \circ \gamma_0\},\end{aligned} \tag{1.37}$$

where

$$h(r,\theta) = g_{\partial\left\{\begin{smallmatrix}0 \le \phi \le 2\pi\theta \\ 0 \le \rho \le r\end{smallmatrix}\right\}} \quad \text{and} \quad \gamma_\theta = g_{\{\rho \to \rho e^{2\pi i\theta}\}}. \tag{1.38}$$

Proof. If g_1 and g_2 are coordinate based gluon potentials for which $\gamma_1 = \gamma_2$, where γ_j is defined as in (1.38), then k defined by

$$k(r,\theta) = g_2^r(r,\theta) \circ (g_1^r(r,\theta))^{-1} \in Mor(P_{(r,\theta)}) \tag{1.39}$$

is actually a gauge transformation for $P \to S^2$. For the equality of the γ's guarantees that $k(1,\theta)$ is independent of θ.

It follows that $(k_* g_1)^r = g_2^r$. If g_1 and g_2 have the same holonomy $h(r,\theta)$ for all r and θ, it then follows that

$$(k_* g_1)^\theta = g_2^\theta, \tag{1.40}$$

because

$$g^\theta(r,\theta) = g^r(r,\theta) \circ h(r,\theta) \circ g^r(r,\theta)^{-1} \tag{1.41}$$

for $g = g_2, k_* g_1$. Thus g_1 and g_2 are equal as based gluons, proving that $[g] \to h$ is 1–1.

Given $\gamma \in P\text{-}component \subset L\,Mor(P_0, P_\infty)$, we can find a continuous field

$$\tilde{\gamma}(r,\theta) \in Mor(P_0, P_{re^{i\theta}}) \tag{1.42}$$

such that $\tilde{\gamma}(0,\theta) = 1$ and $\tilde{\gamma}(1,\theta) = \gamma_\theta$.

Given holonomy $h(r,\theta)$ projecting to γ, we define a gluon potential g by defining $g^r = \tilde{\gamma}$ and using g^r, h and (1.41) to determine g^θ. This shows that $[g] \to h$ is onto.

If we consider all bundle types at once, then we see that

$$\mathcal{G}luons_{based}^{(coord)} = \{h(r,\theta) \in Path^{1,*}Path^{1,*}Mor(P_0)\colon\ h(1,1) = 1\}$$

is a topological group. The projection to the base,

$$h(r,\theta) \to h(1,\theta)$$

is a group homomorphism, and the kernel is the free loop space

$$Path_\theta^{1,*}\,Path_r^{1,1}\,Mor(P_0),$$

which is contractible. Since $\mathcal{G}luons_{based}^{(coord)}(P)$ is a connected component of $\mathcal{G}luons_{based}^{(coord)}$, this implies the topological claims of the proposition. □

Closed surfaces of positive genus. We again first assume that P is the trivial bundle.

Let Σ denote a closed surface of $genus > 0$. Represent Σ via a planar diagram in the usual way (see Fig. 1).

We have located the vertices at the roots of unity, for notational convenience. To distinguish between the two identical copies of α_1, we have labelled one of them α_1'; we do the same thing with β_1, etc. To express relations induced by these identifications, we will need to consider the θ-increments of a function (or section) defined in polar coordinates across the sectors of the disk determined by the rays from the origin to the $4 * genus(\Sigma)$ roots of unity. Thus given

$$\tilde{g}(r,\theta) \in Path^{1,*}Path^{1,*}K, \tag{1.43}$$

we set

$$\tilde{g}^{\alpha_1}(r,t) = \tilde{g}(r, t/(4 * genus)),$$

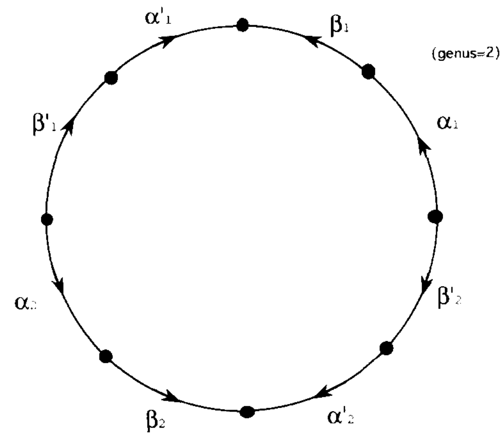

Fig. 1.

$$\tilde{g}^{\beta_1}(r,t) = \tilde{g}\left(r, \frac{t+1}{4*genus}\right)\left(\tilde{g}^{\alpha_1}\left(r, \frac{1}{4*genus}\right)\right)^{-1} \tag{1.44}$$

and so on, where $0 \leq t \leq 1$. The iterated path $\tilde{g}$ and the vector

$$\vec{\tilde{g}} = (\tilde{g}^{\alpha_1}, \ldots, \tilde{g}^{\beta'_{genus}}) \in (Path^{1,*}Path^{1,*}K)^{4*genus} \tag{1.45}$$

clearly determine one another.

As in the preceding cases we first use the map

$$g \to (g^r, g^\theta) \tag{1.46}$$

to obtain isomorphisms

$$\mathcal{P}otentials^{(coord)} \cong D_0K \times F, \tag{1.47}$$

$$\mathcal{G}luons_{based}^{(coord)} \cong D_0K \times_{K_0} F, \tag{1.48}$$

where

$$F = \{\tilde{g} \in Path_r^{1,*}Path_\theta^{1,*}K\colon\ \tilde{g}^{\alpha_1}(1,\cdot) = \tilde{g}^{\alpha'_1}(1,\cdot), \ldots\}. \tag{1.49}$$

This fiber is clearly contractible.

To identify the base in (1.48), suppose $g \in D_0K$, viewed as a purely radial gluon potential, and let $k^{\alpha_1}, k^{\beta_1}, \ldots$ denote the boundary values. For g to be a gauge transformation on Σ applied to the trivial gluon potential, we must have equality along identical edges, i.e.

$$k^{\alpha_1} = k^{\alpha'_1}, \ldots, k^{\beta_{genus}} = k^{\beta'_{genus}}. \tag{1.50}$$

The product

$$\gamma^{\alpha_1} = (k^{\alpha'_1})^{-1}k^{\alpha_1} \tag{1.51}$$

is the holonomy of g around the loop indicated by double arrows in Fig. 2.

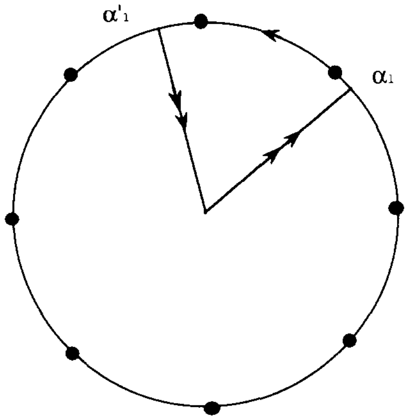

Fig. 2.

This is the first of $2 * genus$ invariants of g modulo $\mathcal{K}_0$, gotten by replacing α_1 in (1.38) by $\alpha_1, \ldots, \beta_{genus}$, respectively; we set

$$\gamma = (\gamma^{\alpha_1}, \ldots, \gamma^{\beta_{genus}}) \in (Path^{*,*}K)^{2*genus}. \tag{1.52}$$

For $\gamma \in (Path^{*,*}K)^{2*genus}$ to be the holonomy of some radial gluon potential, it is necessary that

$$R(\gamma^{\alpha_1}, \gamma^{\beta_1}, \ldots, \gamma^{\beta_{genus}}) = 1, \tag{1.53}$$

where

$$R = \prod_{i=1}^{genus} \gamma^{\beta_i}(0)(\gamma^{\alpha_i}(1))^{-1}(\gamma^{\beta_i}(1))^{-1}\gamma^{\alpha_i}(0), \tag{1.54}$$

and successive terms multiply from the left. For the meaning of the ith term is that it is the holonomy around the path indicated by double arrows in Fig. 3.

Thus R itself is holonomy about the trivial loop. Note that in terms of the $k's$, the ith term of (1.54) is simply

$$(k^{\beta_i'}(0))^{-1}k^{\alpha_i}(0). \tag{1.55}$$

This leads to the following:

Lemma 1.8. *The map* $[g] \to \gamma(g)$ *induces a bijection*

$$D_0K/\mathcal{K}_0 \cong \vec{1}\text{-}component \subset \{\gamma \in (Path^{*,*}K)^{2*genus}\colon\ R(\gamma) = 1\}.$$

If K is connected and simply connected, then $\{\gamma\colon\ R(\gamma) = 1\}$ *is connected.*

Proof. We have already observed that the map is injective. To prove the first statement, it therefore suffices to check that the image of the map is open and closed.

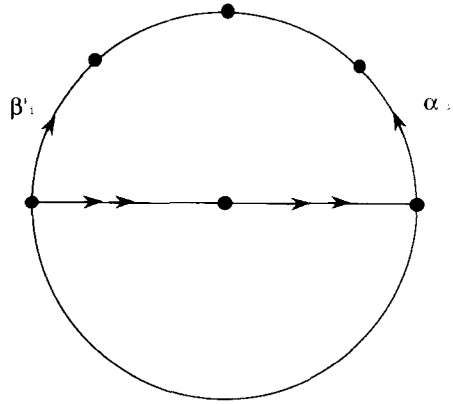

Fig. 3.

The second statement is equivalent to the well-known fact that all K-bundles on Σ are trivial, provided that K is connected and simply connected. But it is worthwhile to prove this directly. Thus we must show that every γ satisfying the relation comes from some radial gluon potential $g \in D_0 K$.

We first determine the boundary values of g. Choose k^{α_1} arbitrarily. This determines $k^{\alpha'_1}$ by (1.55). Since K is connected, we can choose k^{β_1} connecting $k^{\alpha_1}(1)$ and $k^{\alpha'_1}(1)$. This determines $k^{\beta'_1}$, again by (1.55). We then choose k^{α_2} so that it begins at $k^{\beta'_1}(0)$, and so on.

The fact that the relation is satisfied guarantees that when we eventually choose $k^{\beta_{genus}}$, it is then automatically the case that

$$(k^{\beta'_{genus}}(0))^{-1} k^{\alpha_1}(0) = 1,$$

i.e. the boundary values form a continuous loop. The existence of g now follows from the assumption that K is simply connected. □

We now turn to the parameterization of based gluons using holonomy. We will write

$$\vec{h} = (h^{\alpha_1}, \ldots, h^{\beta'_{genus}}) \in (Path^{1,*} Path^{1,*} Mor(P_0))^{4*genus}, \tag{1.56}$$

as in (1.44), where $h^{\delta}(r, t)$ will be interpreted as holonomy about the sector (see Fig. 4) and $\delta = \alpha_1, \ldots, \beta'_{genus}$.

Proposition 1.9. *The space of coordinate based P-gluons has the structure of a fiber bundle with contractible fiber and base space*

$$B = P\text{-}component \subset \{\gamma \in (PathMor(P_0))^{2*genus} \colon R(\gamma) = 1\}.$$

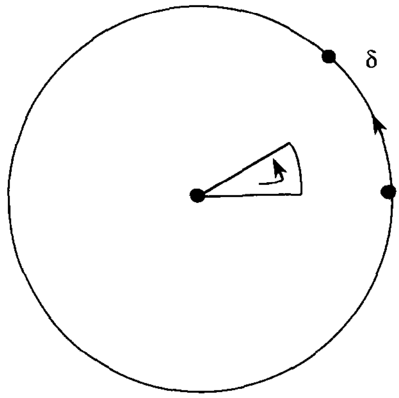

Fig. 4.

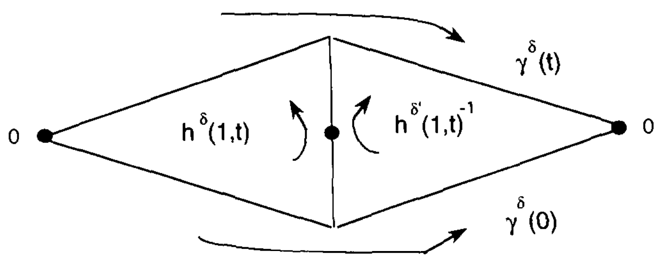

Fig. 5.

More precisely, the map $[g] \to \gamma, \vec{h}$ *induces an isomorphism*

$$\mathcal{G}luon^{(coord)}_{based} \cong \bigsqcup_{\gamma \in B} \Big\{ \vec{h} \in (Path^{1,*}Path^{1,*}Mor(P_0))^{4*genus}:\ \gamma^{\delta}(t) \circ h^{\delta}(1,t) = h^{\delta'}(1,t)^{-1} \circ \gamma^{\delta}(0), \delta = \alpha_1, \beta_1, \ldots, \beta_{genus} \Big\}, \tag{1.57}$$

where h^{δ} *denotes the holonomy of* g *around the contractible sector indicated in Fig. 4, and* γ *is defined as in* (1.52).

Remark 1.10. The geometric meaning of the $2 * genus$ conditions in (1.57) is pictured in Fig. 5.

Proof. Suppose that g_1 and g_2 are two gluon potentials for which $\gamma_1 = \gamma_2$. Define

$$k(r,\theta) = g_2^r \circ (g_1^r)^{-1} \in Mor(P_{(r,\theta)}).$$

This defines an element in $\mathcal{K}_0$ because the γ's are the same.

It then follows that $(k_* g_1)^r = g_2^r$. If g_1 and g_2 also have the same holonomy $\vec{h}$, it then follows that $g_1 = g_2$. This proves that the map is injective.

Now suppose that we are given γ and $\vec{h}$ satisfying the conditions in (1.57). We first must establish the existence of g^r and boundary values $\vec{k}$ such that (1.51) is satisfied. As before we observe that the composition of maps

$$\mathcal{P}otentials^{(coord)} \to C^0\left(\bigsqcup_D Mor(P_0, P_{(r,\theta)})\right) \to C^0\left(\bigsqcup_{S^1} Mor(P_0, P_{(1,\theta)})\right) \to B,$$

$$g \to g^r \to \vec{k} \to \gamma$$

has an image which is open and closed. This follows from the easily established fact that for each of these individual maps, the image of a connected component is open and closed.

Once we have g^r, g^θ is completely determined. The conditions in (1.57) guarantee that the corresponding g is in fact a coordinate based gluon potential on Σ. □

2. The Yang–Mills measure

In two dimensions the Yang–Mills measure can be directly defined as a finite, finitely additive measure on the space of connections, or better, on the space of gluon potentials. We will recall this in Section 2.1. In Section 2.4, by choosing coordinates on the surface, we will identify this measure in terms of conditional probabilities associated to iterates of Wiener measure. We will recall the definition of Wiener measure and its iterates in Sections 2.2 and 2.3, respectively. In Section 2.5 we consider the abelian case, and in Section 2.6 we show that $\pi_* \nu_{YM}$ is supported on $\mathcal{G}luons_{C^{-1-\epsilon}}$, for any $\epsilon > 0$.

2.1. The definition of ν_{YM}

The $Ad\,K$-invariant inner product on $\mathfrak{k}$ induces a bi-invariant Riemannian structure on K. Let Δ denote the corresponding Laplacian, and let

$$H : \mathbb{R}^+ \times K \to \mathbb{R}^+ \colon\ t, g \to H_t(g) \tag{2.1}$$

denote the corresponding convolution heat kernel, i.e.

$$((\partial/\partial t) - \tfrac{1}{2}\Delta)H = 0 \quad \text{and} \quad \lim_{t\downarrow 0} H_t = \delta_1. \tag{2.2}$$

Then

$$H_t(g)\,\mathrm{d}g \in Prob(K) \quad \text{and} \quad H_s * H_t = H_{s+t}, \tag{2.3}$$

where $\mathrm{d}g$ denotes the invariant probability on K.

Given a tiling $\mathcal{T}$ of Σ with oriented edges, let V, E and F denote the sets of vertices, oriented edges and faces, respectively. There is a natural projection

$$\pi_{\mathcal{T}} : \mathcal{G}luons \to \prod_E Mor(P_{e_0}, P_{e_1})\colon\ g \to (g_e)_{e\in E}, \tag{2.4}$$

where g_e denotes parallel translation from the initial fiber to the initial fiber along the oriented edge e.

Define a measure on the space $\prod Mor(P_{e_0}, P_{e_1})$ by

$$d\nu_{\mathcal{T}} = \prod_{f \in F} H_{area(f)}(g_{\partial f}) \prod_{e \in E} dg_e, \tag{2.5}$$

where $g_{\partial f}$ is the "holonomy" around the boundary of f, i.e. $g_{\partial f}$ represents the conjugacy class of the morphism

$$g_{e_n}^{\epsilon_n} \circ \cdots \circ g_{e_1}^{\epsilon_1} \in Mor(P_{basepoint}), \tag{2.6}$$

where

$$\partial f = \pm(\epsilon_1 e_1 + \cdots + \epsilon_n e_n). \tag{2.7}$$

Since the holonomy is a conjugacy class, we can view it as a conjugacy class in K (which is noncanonically isomorphic to $Mor(P_q)$, for each q).

Suppose that $\mathcal{T}'$ is a refinement of $\mathcal{T}$. There is then a natural projection

$$p : \prod_{E'} Mor \to \prod_{E} Mor. \tag{2.8}$$

The main result is the following, which is traced to Migdal in [Wi1].

Proposition 2.1. $p_*(\nu_{\mathcal{T}'}) = \nu_{\mathcal{T}}$.

Proof. A refinement involves either splitting edges or connecting vertices across faces, or both. Suppose first that $\mathcal{T}'$ is obtained by splitting an edge $e \in E$ as $e = e_2 \circ e_1$. Then the projection p sends g_{e_1} , g_{e_2} to g_e, where

$$g_e = g_{e_2} \circ g_{e_1}. \tag{2.9}$$

Since the integrand for $\nu_{\mathcal{T}'}$ depends upon g_{e_1} and g_{e_2} only through g_e, consistency is obvious in this case.

Suppose now that $\mathcal{T}'$ is obtained by subdividing a face $f \in F$ into two faces $f_1, f_2 \in F'$, by inserting an oriented edge e'. We must show that

$$\int H_{Area(f_2)}(g_{\partial f_2}) H_{Area(f_1)}(g_{\partial f_1})\, dg_{e'} = H_{Area(f)}(g_{\partial f}). \tag{2.10}$$

This follows from the convolution formula (2.3), because

$$g_{\partial f} = g_{\partial f_2} \circ g_{\partial f_1}, \tag{2.11}$$

and

$$g_{\partial f_1} = g_{e'}^{-1} \circ k, \qquad g_{\partial f_2} = h \circ g_{e'}, \tag{2.12}$$

for some fixed morphisms k and h. □

Let $\mathcal{B}$ denote the subalgebra of the Borel subsets of $\mathcal{P}otentials$ generated by the mappings $\pi_{\mathcal{T}}$, for all tilings $\mathcal{T}$.

Corollary 2.2. *There is a finitely additive measure ν_{YM} on $(\mathcal{P}otentials, \mathcal{B})$, with finite total measure, with the property that ν_{YM} projects to $\nu_{\mathcal{T}}$, relative to the map $\pi_{\mathcal{T}}$, for each tiling $\mathcal{T}$. This measure is invariant with respect to the natural action of $Aut(P)_{\omega}$.*

This construction is natural with respect to sewing. To formulate this, suppose that the surface $\check{\Sigma}$ is obtained from a surface Σ by gluing one boundary component of Σ to another. Suppose also ω maps to $\check{\omega}$, with respect to the natural map $\Sigma \to \check{\Sigma}$. There are natural maps

$$\begin{aligned} &Paths\,\Sigma \to Paths\,\check{\Sigma}, \\ &\mathcal{P}otentials(\Sigma) \leftarrow \mathcal{P}otentials(\check{\Sigma}). \end{aligned} \tag{2.13}$$

The functorial properties of the measure ν_{YM} are summarized by the following:

Corollary 2.3. *With respect to the map* (2.13),

$$\nu_{YM,\Sigma} \leftarrow \nu_{YM,\check{\Sigma}}.$$

Also if Σ is the disjoint union of two surfaces with volume elements, then $\nu_{YM,\Sigma}$ is simply the corresponding product measure.

2.2. Wiener measures

Let T denote a positive constant, thought of as temperature. The Wiener measure ν_T on the path space $Path^{1,*}K$ is defined as follows. Fix a subdivision (or triangulation) of the unit interval, $0 = s_0 < s_1 < \cdots < s_n = 1$. This determines a projection

$$\pi_{\bar{s}} : Path^{1,*}K \to K^n\colon\ g \to (g(s_j))_{1 \le j \le n}. \tag{2.14}$$

We then define a probability measure on the image K^n by

$$\begin{aligned} \pi_{\bar{s}*}\nu_T &= \prod_1^n H_{T(s_i - s_{i-1})}(g_{i-1}^{-1} g_i) \prod_1^n \mathrm{d}g_i \\ &= \prod_1^n H_{T(s_i - s_{i-1})}(g_{i-1}^{-1} g_i)\, \mathrm{d}(g_{i-1}^{-1} g_i). \end{aligned} \tag{2.15}$$

These projections consistently define a finitely additive measure $\nu_T = \nu_T^{1,*}$ on the path space, because of the semigroup property (2.3). It is a nontrivial fact that ν_T has a countably additive extension.

In terms of the principal fibration

$$Path^{1,*}K \to K\colon\ g \to g(1), \tag{2.16}$$

we have the disintegration

$$\mathrm{d}\nu_T^{1,*} = \int_G \nu_T^{1,h} H_T(h)\,\mathrm{d}h, \tag{2.17}$$

where $\nu_T^{1,h}$ is the normalized conditional Wiener measure on the path space $Path^{1,h}K$. The probability measure $\nu_T^{1,h}$ is determined by its projection with respect to the restriction of the map in (2.14), where n is replaced by $n-1$. The image is defined by the expression in (2.15), when g_n is replaced by h, and the entire expression is divided by $H_T(h)$.

The normalized Brownian bridge measure $\nu_T^{1,1}$ on ΩK has positive measure on each connected component. Relative to the fibration

$$0 \to (\Omega K)_0 \to \Omega K \xrightarrow{\pi} \pi_1 K \to 0, \tag{2.18}$$

we have

$$\pi_* \nu_T^1 = \sum_{\gamma\in\pi_1} c_\gamma \delta_\gamma \quad \text{and} \quad \nu_T^1 = \int_{\pi_1} \nu_T^\gamma c_\gamma \,\mathrm{d}(\pi_*\nu_T^1)(\gamma), \tag{2.19}$$

where

$$c_\gamma > 0, \tag{2.20}$$

and ν_T^γ denotes the normalization of the restriction of the Brownian bridge to the γ-connected component of ΩK. The numbers c_γ can undoubtedly be computed explicitly. In fact it is interesting to ask, given an arbitrary compact Riemannian manifold with basepoint (M,m) relative to the projection

$$\Omega(M,m) \xrightarrow{\pi} \pi_1(M,m),$$

what can one say about the numbers c_γ for the projection

$$\pi_* \nu_T^m = \sum_{\gamma\in\pi_1} c_\gamma \delta_\gamma,$$

where ν_T^m is the Brownian bridge?

2.3. *Iterates of Wiener measure*

As we pointed out above, the abstract fact underlying the existence of Wiener measure is the semigroup property. The family of Wiener measures $\{\nu_T\}_{0<T}$ also has the semigroup property, on the group $Path^{1,*}K$:

$$\nu_{t_1} * \nu_{t_2} = \nu_{t_1+t_2}. \tag{2.21}$$

This follows from the easily verified fact that the one-parameter family of projections $\{\pi_{\bar{s}*}\nu_t\colon t>0\}$, defined by (2.15), is a semigroup (see [M]).

This makes it possible to iterate the Wiener construction. Fix T as before. Given a subdivision as before, there is a projection

$$\Pi_{\bar{s}*} : Path^{1,*}Path^{1,*}K \to (Path^{1,*}K)^n\colon\ \gamma \to (\gamma(s_i)). \tag{2.22}$$

We define a probability measure on $(Path^{1,*}K)^n$ by

$$\Pi_{\vec{s}*}\nu_T^{(2)} = \prod_1^n \mathrm{d}\nu_{(s_i - s_{i-1})T}(g_{i-1}^{-1}g_i). \tag{2.23}$$

The semigroup property (2.21) implies that the projections (2.23) consistently define a finitely additive measure $\nu_T^{(2)}$ on the double path space. It can be proved that the measure $\nu_T^{(2)}$ has a countably additive extension, by the same argument as for Brownian motion on a finite-dimensional Lie group.

If we choose a second subdivision $0 < t_1 < \cdots < t_m = 1$, and we set $g_{ij} = \gamma(s_i, t_j)$, then the projection $\gamma \to (g_{ij})$ maps $\nu_T^{(2)}$ to

$$\prod_{i=1}^n \prod_{j=1}^m H_{(s_i - s_{i-1})(t_j - t_{j-1})T}(g_{i,j-1}^{-1}g_{i-1,j-1}g_{i-1,j}^{-1}g_{i,j})\,\mathrm{d}g_{i,j}. \tag{2.24}$$

This follows directly from (2.15) and (2.23).

In this section we will consider two conditional disintegrations of the measure $\nu_T^{(2)}$. First, relative to the principal fibration

$$Path^{1,*}(Path^{1,*}K) \to Path^{1,*}K\colon\ \gamma \to \gamma(1), \tag{2.25}$$

there is a disintegration

$$\mathrm{d}\nu_T^{(2)} = \int_{Path^{1,*}K} \nu_T^{(2)}(g|g(1) = h)\,\mathrm{d}\nu_T(h), \tag{2.26}$$

where $h \to \nu_T^{(2)}(g|g(1) = h)$ is the associated regular conditional probability distribution on the fibers, $Path^{1,h}(Path^{1,*}K)$. These conditional measures can be described in a quasi-explicit way using Ito's isomorphism of probability spaces

$$I : (Path^{0,*}\mathfrak{k}, \nu_T^{0,*}) \to (Path^{1,*}K, \nu_T^{1,*})\colon\ x \to g, \tag{2.27}$$

where x and g are related by the stochastic differential equation, interpreted in the sense of Fisk–Stratonovich,

$$\mathrm{d}x = g^{-1} \circ \mathrm{d}g. \tag{2.28}$$

We will simply state the result, since Ito's isomorphism reduces the problem to a linear problem.

Proposition 2.4. *We have*

$$\pi_{\vec{s}*}\nu^{(2)}(g_1, \ldots, g_n|g_n = h)$$
$$= \mathrm{d}\nu_{T\Delta_2(s_1/s_2)}(\tilde{g}_2^{-1}g_1)\,\mathrm{d}\nu_{T\Delta_3(s_2/s_3)}(\tilde{g}_3^{-1}g_2)\cdots\mathrm{d}\nu_{T\Delta_n(s_{n-1}/s_n)}(\tilde{g}_n^{-1}g_{n-1}),$$

where $\Delta_m = s_m - s_{m-1}$, *and* $\tilde{g}_l = I((s_{l-1}/s_l)I^{-1}g_l)$.

The second conditioning is more elementary. For the fibration

$$Path^{1,*}Path^{1,*}K \to K\colon\ g \to g(1,1) \tag{2.29}$$

there is a disintegration

$$\nu_T^{(2)} = \int_K \{\nu_T^{(2)}(\cdot|g(1,1)=k)\}H_T(k)\,\mathrm{d}k. \tag{2.30}$$

In this case, as in the one-dimensional case, we can explicitly construct the conditional probabilities; for the projection corresponding to (2.24), the image of $\nu_T^{(2)}(\cdot|g(1,1)=k)$ is given by the formula (2.24), where $g_{n,n}$ is set equal to k, $\mathrm{d}g_{n,n}$ is deleted and the entire expression is divided by $H_T(k)$.

2.4. YM_2 and iterates of Wiener measure

By considering the path space representations for coordinate based gluons in Section 1.3, we will now show that $\pi_*\nu_{YM}$ can be realized as a conditional measure for 2-fold iterates of Wiener measure. Our goal is to show that this can be done in principle, and we will work through all of the details only in the cases of the disk and closed oriented surfaces.

The disk. As an elementary example we first consider the disk with Lebesgue measure. We will use the parameterization by holonomy

$$\mathcal{G}luons_{based}^{(coord)} \to Paths^{1,*}Paths^{1,*}K\colon\ g \to (g_{r,\theta}) = \left(g_{\partial\left\{\begin{smallmatrix}0\le\phi\le 2\pi\theta\\ 0\le\rho\le\sqrt{r}\end{smallmatrix}\right\}}\right). \tag{2.31}$$

Take subdivisions $0=r_0<\cdots<r_n=1$ and $0<\theta_0<\cdots<\theta_n=1$. This gives rise to a tiling $\mathcal{T}$ of the disk, and the corresponding projection of ν_{YM} is given by

$$\begin{aligned}\pi_*^{\mathcal{T}}\nu_{YM} &= \prod H_{\pi(\theta_i-\theta_{i-1})(r_j-r_{j-1})}\left(g_{\partial\left\{\begin{smallmatrix}2\pi\theta_{i-1}\le\phi\le 2\pi\theta_i\\ \sqrt{r_{j-1}}\le\rho\le\sqrt{r_j}\end{smallmatrix}\right\}}\right)\prod_{Edges}\mathrm{d}g_e\\ &= \prod H_{\pi(\Delta\theta)(\Delta r)}(g(r_j,\theta_i)g(r_j,\theta_{i-1})^{-1}g(r_{j-1},\theta_{i-1})g(r_{j-1},\theta_i)^{-1})\prod\mathrm{d}g_e\end{aligned} \tag{2.32}$$

When we project from $\{g_e\}$ to the holonomy variables $\{g(r_j,\theta_i)\}$ the integrand in (2.32) does not change. The resulting expression is exactly the corresponding local expression (2.24) for the iterated Wiener measure $\nu_\pi^{(2)}$. This proves the following:

Proposition 2.5. *Relative to the identification* (2.31),

$$\pi_*\nu_{YM}^{Leb} = \nu_\pi^{(2)}.$$

The sphere. We equip the sphere S^2 with the standard area form. In order to neatly match up YM_2 with an iterated path measure, we parameterize S^2 using the disk in such a way that

$$Area(\{r \le \rho \le r + \Delta r, \theta \le \psi \le \theta + \Delta\theta\}) = 4\pi(\Delta\theta)(\Delta r). \tag{2.33}$$

This can be done by starting with geodesic coordinates and reparameterizing the radial coordinate (explicitly, $\phi = \frac{1}{2}(1 - \cos(r))$, where ϕ is the azimuthal angle).

As in Proposition 1.7, we can use holonomy to identify $\mathcal{G}luons_{based}^{(coord)}$ with

$$\{h(r, \theta) \in Path^{1,*}Path^{1,*}Mor(P_0)\colon\ h(1, \theta) \in [P] \subset \Omega\ Mor(P_0)\}, \tag{2.34}$$

where $[P]$ denotes the connected component corresponding to P. This is a fiber bundle with base $[P] \subset \Omega\ Mor(P_0)$, where the projection is given by

$$proj : h \to \eta, \qquad \eta(\theta) = h(1, \theta). \tag{2.35}$$

For our purposes it is useful to think of the total space as the open subset $proj^{-1}([P])$ of the inverse image of 1 with respect to the projection (2.27).

As in the case of the disk, we consider the projection $\pi_*^T \nu_{YM}$. This is given by the same formula (2.32), except that π is replaced by 4π, and $g(r_n, \theta_n)$ is replaced by 1. By comparing this modified version of (2.32) and the modified version of (2.24), which represents the conditional measure (for $k = 1$) in (2.30), we conclude that

$$\pi_*\nu_{YM} = H_{4\pi}(1) * (1/E)\nu_{4\pi}^{(2)}(\cdot|h(1, 1) = 1)|_{proj^{-1}([P])}. \tag{2.36}$$

By considering the regular conditional disintegration associated to the fiber bundle structure, we obtain the following abstract.

Proposition 2.6. *In reference to* (2.34), *we have*

$$\pi_*\nu_{YM} = H_{4\pi}(1) \int_{[P]} \nu_{4\pi}^{(2)}(g|g(1) = \eta)\, \mathrm{d}\nu_{4\pi}^{[P]}(\eta).$$

Remark 2.7. The integrand is the regular conditional distribution for the iterated Wiener measure on the spaces

$$Path^{1,\eta}(Path^{1,*}Mor(P_0)),$$

as defined in (2.26), and which we described quasi-explicitly in Proposition 2.4; $\mathrm{d}\nu_{4\pi}^{[P]}(\gamma)$ denotes the normalized restriction of the Brownian bridge to the P-component of $\Omega\ Mor(P_0)$, as in (2.19).

Closed surfaces of genus > 0. As in the case of the sphere, we parameterize the surface Σ (minus a 1-simplex) by a disk, where area is computed by (2.33), with the total area T in place of 4π. We will first identify $\pi_*\nu_{YM}$ with a conditional measure on the space of conditioned multiple paths in Proposition 1.9.

According to Proposition 1.9, we can identify $\mathcal{G}luons_{based}^{(coord)}$ as a total space in the following way. There is a projection

$$\begin{aligned} &\bar{R} : (Path^{1,*}Path^{1,*}Mor(P_0))^{4*genus} \times Mor(P_0)^{2*genus} \to K, \\ &\bar{R}((\vec{h}, \gamma_0)) = R(\gamma), \end{aligned} \tag{2.37}$$

where $\gamma = proj(\vec{h}, \gamma_0)$ is determined by the initial conditions $\gamma^\delta(0) = \gamma_0^\delta$ and the $2*genus$ conditions in (1.57), and R is defined by (1.54). Then $\mathcal{G}luons_{based}^{(coord)}$ is the open subset $proj^{-1}([P])$ of the inverse image of 1 with respect to $\bar{R}$.

Equip the space

$$(Path^{1,*}Path^{1,*}Mor(P_0))^{4*genus} \times Mor(P_0)^{2*genus} \tag{2.38}$$

with the probability μ equal to the product of the iterated Wiener measures, each with temperature $T/(4*genus)$, and the invariant probabilities on the factors $Mor(P_0)$. We assert that

$$\pi_*\nu_{YM} = const * (1/E)\mu(\cdot|\bar{R} = 1)|_{proj^{-1}([P])}. \tag{2.39}$$

To understand the meaning of the right-hand side, note that

$$\bar{R}(\vec{h}, \gamma_0) = \prod\{\gamma_0^\beta h^\alpha(1,1)(\gamma_0^\alpha)^{-1}h^{\alpha'}(1,1)h^\beta(1,1)(\gamma_0^\beta)^{-1}h^{\beta'}(1,1)\gamma_0^\alpha\}. \tag{2.40}$$

All the distributions involved in this formula are smooth, so that the distribution of $\bar{R}$ is a smooth function times Haar measure. Hence we can define the conditional measure $\mu(\cdot|\bar{R} = 1)$, as we did in the last paragraph of Section 2.3.

To check (2.39), as in the previous cases, we first consider $\pi_*^{\mathcal{T}}\nu_{YM}$, where we assume that $\mathcal{T}$ includes the $4*genus$ roots of unity as vertices. This will again be given by a formula of the form (2.32), where the integrand depends only upon the holonomy variables. When we project from the edge group variables to the holonomy variables, we obtain (2.39). The constant has to be computed independently, say as in [Wi1].

This leads to the following:

Proposition 2.8. *Relative to the projection*

$$proj : \mathcal{G}luons_{based}^{(coord)} \to [P] \subset \{\gamma \in (Path^{*,*}Mor(P_0))^{2*genus}: R(\gamma) = 1\},$$

there is a regular conditional disintegration

$$\pi_*\nu_{YM} = const \int_{[P]} \{\nu_\gamma\}\, d\nu^{[P]}(\gamma|R = 1),$$

where $\nu^{[P]}(\gamma|R = 1)$ denotes the conditioned measure for the product of Wiener measures

$$\left((Path^{*,*}Mor(P_0))^{2*genus}, \prod \nu_{T/(2*genus)}\right).$$

2.5. The abelian case

Suppose that Σ is closed and $K = T$, a torus. In the case of S^2, the simply connected case, $\pi_*\nu_{YM}$ (as a finitely additive measure) is essentially a Gaussian cylinder measure, in terms of curvature (in probabilistic jargon, the curvature is identified with white noise for the conditioned $\mathfrak{t}$-valued Brownian sheet that projects to $\pi_*\nu_{YM}$ via the exponential map). The nonsimply connected cases are only slightly more complicated; in these cases the space of gluons is a $H^1(\Sigma, T)$-principal bundle, $\pi_* \nu_{YM}$ is $H^1(\Sigma, T)$-invariant, and the projection to the base (gluons restricted to contractible curves) can be expressed in terms of a Gaussian distribution of the curvature. All this is well known, but it is worth reviewing it from our point of view.

We begin with a few preparatory remarks. The exponential map defines an exact sequence

$$0 \to \Lambda \to \mathfrak{t} \to T \to 0. \tag{2.41}$$

For the corresponding exact sequence of sheaves of C^∞-functions on Σ with values in these groups, the induced long exact sequence of cohomology yields the first Chern class classification of principal T-bundles,

$$H^1(\Sigma, C_T^\infty) \cong H^2(\Sigma, \Lambda) \cong \Lambda. \tag{2.42}$$

Suppose that A is a T-connection on the T-bundle $\pi : P \to \Sigma$. Because T is abelian the curvature $\mathrm{d}A$ has a well-defined push-forward $\pi_* \,\mathrm{d}A$. The image of the map

$$\mathcal{A} \to \Omega^2(\Sigma, \mathfrak{t}) : A \to F, \quad \text{where } F = \pi_* \,\mathrm{d}A, \tag{2.43}$$

is the affine submanifold defined by the condition

$$\int_\Sigma F = c_1(P) \in \Lambda. \tag{2.44}$$

The case $\Sigma = S^2$. The way in which the Gaussian distribution of curvature, the Brownian sheet, and $\pi_* \nu_{YM}$ are related is summarized by the following diagram:

$$\begin{array}{ccc}
\left\{F \in \Omega^2(\Sigma, \mathfrak{t}) : \int_\Sigma F = c_1(P)\right\} & \xleftarrow{\mathrm{d}} & \mathcal{A} \\
\updownarrow & & \downarrow (holonomy) \\
\left\{x \in Path^{0,*} Path^{0,*}\mathfrak{t} : \ x(1,1) = c_1(P)\right\} & \xleftrightarrow{\exp} & \{h \in Path_r^{1,*} Path_\theta^{1,*} T : \ h(1,\cdot) \in [P]\} \\
 & & \uparrow \\
 & & \mathcal{G}luons(P)
\end{array} \tag{2.45}$$

Here x is the normalized $\mathfrak{t}$-valued Brownian sheet, corresponding to the inner product $\langle \cdot, \cdot \rangle$, with the indicated conditioning; h is distributed according to $\pi_* \nu_{YM}$; and F is a Gaussian centered at $*c_1(P)$, corresponding to the inner product

$$\int_\Sigma \langle F_1 \wedge *F_2 \rangle. \tag{2.46}$$

The relation between x and F is given by

$$x(r, \theta) = \int_0^r \int_0^\theta F, \quad \text{or} \quad F = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}(x)\right) \mathrm{d}r \wedge \mathrm{d}\theta. \tag{2.47}$$

For later purposes it is useful to note that the correspondences in (2.45) are natural with respect to automorphisms. We can exploit this to compute the natural representation. Let H_0 denote the Hilbert space completion of

$$\left\{ F \in \Omega^2(\Sigma, \mathfrak{t}) : \int_\Sigma F = 0 \right\} \tag{2.48}$$

in the norm given by (2.46). Let ν_G denote the corresponding Gaussian cylinder measure. Then as $SDiff(\Sigma)$-modules

$$L^2(\pi_* \nu_{YM}) \cong L^2(\mathrm{d}\nu_G(\cdot + *c_1(P))) \cong L^2(\mathrm{d}\nu_G) \cong \sum_{k \geq 0} \oplus S^k(H_0^*), \tag{2.49}$$

where the last correspondence is defined by the Fourier transform (see Ch. 6 of [Hi]). In the case that $\mathfrak{t}$ is one dimensional, the components of the decomposition (2.49) are irreducible (this is originally due to Kirillov; see [VGG]). The most transparent proof of this uses the fact that $SDiff(\Sigma)$ is n-fold transitive on Σ (see [Ch]).

The positive genus case. Consider the relations which define the space of gluons in the case of positive genus, as described in the paragraph following (2.37). Since K is abelian, the relation $\bar{R} = 1$ in (2.40) is equivalent to

$$\prod h^\alpha(1,1) h^{\alpha'}(1,1) h^\beta(1,1) h^{\beta'}(1,1) = \left(\prod \gamma_0^\beta (\gamma_0^\alpha)^{-1} (\gamma_0^\beta)^{-1} \gamma_0^\alpha \right)^{-1}. \tag{2.50}$$

The group element on the left is the identity. Thus both sides of (2.50) are the identity.

Recall that the γ_0^δ are parallel translation around the closed paths described in Fig. 2, with $\theta = 0$, which we will denote by $\delta^\perp$. These paths are generators for the fundamental group of Σ based at the center of the disk. That the right-hand side of (2.50) is 1 then says that the γ_0^δ define a representation of $\pi_1(\Sigma, 0)$.

The fact that the left-hand side of (2.50) is 1 implies that the map

$$r \to \prod h^\alpha(r,1) h^{\alpha'}(r,1) h^\beta(r,1) h^{\beta'}(r,1) \tag{2.51}$$

is in ΩT, the based loop space. The condition $\gamma \in [P]$ says that this loop has winding number $c_1(P)$.

We thus see that the space of gluons is a product,

$$\mathcal{G}luons^{(coord)} \cong H^1(\Sigma, T) \times \{[g] : g_{\delta^\perp} = 1, \delta = \alpha_1, \ldots, \beta_{genus}\}, \tag{2.52}$$

where we identify $H^1(\Sigma, T)$ with based gluons that depend only upon the homotopy type of closed paths. It is not hard to see that $\pi_* \nu_{YM}$ is a product measure in this presentation. However this is not quite what we want, because the product structure (2.52) is not $SDiff(\Sigma)$-equivariant.

Instead we will view $\mathcal{G}luons$ and $\mathcal{G}luons^{(coord)}$ as $H^1(\Sigma, T)$-principal bundles,

$$\begin{array}{l} \mathcal{G}luons \leftarrow H^1(\Sigma, T) \\ \quad\downarrow \\ \quad\mathcal{B} \end{array} \tag{2.53}$$

The action of $H^1(\Sigma, T)$ on gluons is given by

$$\mathcal{G}luons \times H^1(\Sigma, T) \to \mathcal{G}luons: g, k \to g \cdot k, \tag{2.54}$$

where $(g \cdot k)_c = g_c k_{[c]}$, for a closed path c. Here we are implicitly using the fact that $H^1(\Sigma, T)$ can be defined without reference to a basepoint, and $\mathcal{G}luons$ can be realized as functors on closed paths independent of basepoint, as in Section 1.2, because T is abelian.

The base space $\mathcal{B}$ can be identified with functors on the involutive subcategory of contractible paths, where the projection is simply given by restriction. Intuitively these are gluons whose values can be expressed in terms of curvature alone.

To realize the projection $\pi_* \nu_{YM}$ to the base $\mathcal{B}^{(coord)}$, we can use an analogue of the diagram (2.45), where the condition on the curvature F remains the same (the integral over Σ is $c_1(P)$), and the double path space with values in the Lie algebra is replaced by $4 * genus$ copies of the path space, and the constraint

$$\sum_1^{genus} (x_\alpha(1,1) + x_{\alpha'}(1,1) + x_\beta(1,1) + x_{\beta'}(1,1)) = c_1(P) \tag{2.55}$$

is imposed.

Proposition 2.9. *Viewing $\mathcal{G}luons$ as a $H^1(\Sigma, T)$-principal bundle, as in* (2.64), *we have*

$$\pi_* \nu_{YM} = const \int_{\mathcal{B}} \{\nu_{Haar}\} \, \mathrm{d}\pi_* \nu_G.$$

2.6. Support properties of $\pi_* \nu_{YM_2}$

Given that we have identified $\pi_* \nu_{YM_2}$ with a conditional version of iterated Wiener measure, we will now use ideas from [Dudley] to argue that $\pi_* \nu_{YM_2}$ is supported on the space of gluons dual to singular C^α-knots, $\alpha > 1$ (see Remark 1.2(4)). Our argument is a complete proof only in the abelian case.

Conjecture 2.10. *The finitely additive measure $\pi_* \nu_{YM}$ extends uniquely to a countably additive Borel measure on the space of gluons dual to singular C^α-knots, for each $\alpha > 1$.*

Proof. For *abelian K*. We first consider the disk. In this case the result follows almost directly from Theorem 4.2 of [Dudley], as we will now explain.

Suppose that c is a closed embedded loop in Σ (of some smoothness class). The random variable associated to c by holonomy h_c is the exponential of the Gaussian variable associated to the characteristic function $\chi_{Int(c)}$, where $Int(c)$ is the region of Σ interior to c (this is well defined because c is oriented); formally,

$$h_c : F \to \exp\left(\int_{Int(c)} F \right). \tag{2.56}$$

We will write x_c for the random variable

$$x_c : F \to \int_{Int(c)} F. \tag{2.57}$$

Dudley's theorem implies that if the space of C^α-closed embedded curves, $\mathcal{K}nots^{*=*}_{C^\alpha}(\Sigma)$, is equipped with the metric

$$\rho(c_1, c_2) = \omega(Int(c_1) \Delta Int(c_2)) \tag{2.58}$$

(the measure of the symmetric difference) then for each $c \in \mathcal{K}nots^{*=*}_{C^\alpha}(\Sigma)$, there is a version of h_c such that

$$c \to h_c(F) \tag{2.59}$$

is a continuous function with probability one, provided that $\alpha > 1$. In fact combining Theorems 4.2 and 2.1 of [Dudley], and Theorem 3.1 in [Dudley2], we can assert that there is a modulus of continuity for $x_c(F)$ on bounded sets; to be precise, given s such that $\alpha^{-1} < s < 1$,

$$|x_{c_1}(F) - x_{c_2}(F)| \le K\rho(c_1, c_2)^{1-s} \tag{2.60}$$

for all c_j such that $|c_j|_{C^\alpha} \le M$, with probability one, where K depends on M and s. [This is deduced from Dudley's results in the following way. The set of knots with C^α-norm bounded by M is contained in the set $C = I(2, \alpha, M)$, in Dudley's notation. Theorem 3.1 of [Dudley2] asserts that

$$\epsilon^{-t} < H(C, \epsilon) < \epsilon^{-s}$$

for small ϵ and for any $t < \alpha^{-1} < s$, where $H(C, \epsilon)$ is the metric entropy of C. Theorem 2.1 of [Dudley] then asserts that the two functions of h,

$$\int_0^h H(C, \epsilon)\, d\epsilon < \frac{1}{1-s} h^{1-s},$$

are both moduli of continuity.]

If c is a singular knot, then x_c can be written as a linear combination of random variables of the form (2.57). It then follows from (2.58) that

$$|x_{c_1}(F) - x_{c_2}(F)| \le K|c_1 - c_2|_{C^\alpha}^{1-s} \tag{2.61}$$

for all singular knots with $\le d$ self-intersections and $|c_j|_{C^\alpha} \le M$, with probability one, where K depends upon M, d and s. This implies continuity in the abelian case. □

We now consider the nonabelian case. Let $h(r, \theta) \in Path_r^{1,*} Path_\theta^{1,*} K$ denote holonomy around the pie shaped region as in Fig. 5. To express holonomy around a general curve based at zero, we will use the radial gauge

$$g^r = 1, \quad g^\theta = h. \tag{2.62}$$

Let $\partial_2 h$ denote the differential of h with respect to the second variable. Then the stochastic process

$$x(r,\theta) = \int_0^\theta h^{-1}(r,\psi) \circ \partial_2 h(r,\psi), \tag{2.63}$$

interpreted in the sense of Fisk-Stratonovich, is the Brownian sheet with values in $\mathfrak{k}$ (formally, the connection is given by

$$A = \frac{\partial x}{\partial \theta}\,(\mathrm{d}\theta). \tag{2.64}$$

To express holonomy in terms of x, we will use line integrals as defined in Ch. 6 of [Wa] (see also Section 3 of [Dr]). Given a singular knot c based at 0, parallel translation along c is the solution of the stochastic integral equation

$$g_c(t) = 1 + \int_0^t g_c(\tau) \circ \partial_2 x(c(\tau)). \tag{2.65}$$

(To make sense of the stochastic integral, it is necessary to divide the curve c into pure pieces, in the terminology of Walsh; this is possible because c is a singular knot.) We then have $h_c = g(c,1)$, where

$$g(c,t) = \sum_{n\geq 0} \eta_n(c,t), \tag{2.66}$$

$$\eta_0(c,t) = 1, \qquad \eta_{n+1}(c,t) = \int_0^t \eta_n(c,\tau) \circ \partial_2 x(c(\tau)). \tag{2.67}$$

Thus

$$h_c = 1 + x_c + \int_0^1 \left\{ \int_0^{\tau_1} \partial_2 x(c(\tau_2)) \right\} \circ \partial_2 x(c(\tau_1)) + \cdots, \tag{2.68}$$

where x_c is the continuous extension to C^α-curves discussed in the abelian case.

We now need to show that we can choose each η_n to be a continuous function of both c and t. The following conjectural lemma would complete the proof of Conjecture 2.10.

Conjectured lemma 2.11. *We have*

$$|\eta_n(c_1,t) - \eta_n(c_2,t)| \leq K^n \rho(c_1,c_2)^n t^n$$

for all c_j such that $|c_j|_{C^\alpha} \leq M$, with probability one.

Idea of proof. This is true for η_1, based on our knowledge of the abelian case, because

$$\eta_1(c,t) = x_{\tilde{c}},$$

where $\tilde{c}$ is the closed curve which traces c to time t, then swings around to the line $\theta = 0$ along a circle, and returns to the origin; this is because in our gauge parallel translation is trivial along the latter two segments.

3. On harmonic analysis

3.1. On decomposing $SDiff \times L^2(\pi_* \nu_{YM})$

The finitely additive measure $\nu_{YM} = \nu_{YM}^{\omega}$ is invariant with respect to the natural action

$$Aut(P, \omega) \times \mathcal{P}otentials \to \mathcal{P}otentials. \tag{3.1}$$

In two dimensions the natural map $Aut(P) \to Diff(\Sigma)$ is surjective, for any P. It follows that the finitely additive measure $\pi_* \nu_{YM}$ is invariant with respect to the induced action

$$SDiff(\Sigma) \times \mathcal{G}luons \to \mathcal{G}luons. \tag{3.2}$$

Given $c_1, \ldots, c_n \in Path^{p,p}(\Sigma)$ and a function $f : K^n \to \mathbb{C}$ which is K-conjugation invariant, there is an associated function

$$\Phi_{c,f} : \mathcal{G}luons \to \mathbb{C}: [g] \to f(g_{c_1}, \ldots, g_{c_n}). \tag{3.3}$$

This makes sense because $Mor(P_p)$ and K are canonically isomorphic modulo inner automorphisms. For $\phi \in SDiff(\Sigma)$

$$\phi_* \Phi_{c,f} = \Phi_{\phi \circ c, f}. \tag{3.4}$$

Now consider the unitary representation

$$SDiff(\Sigma) \times L^2(\mathcal{G}luons, \pi_* \nu_{YM}). \tag{3.5}$$

There is an invariant filtration of a dense subspace of this L^2-space,

$$(L^2)^{(0)} \subset (L^2)^{(1)} \subset \cdots, \tag{3.6}$$

where $(L^2)^{(n)}$ is spanned by L^2-functions of the form (3.3). Let

$$L^2 = \sum_{n \geq 0} \oplus L^2_{(n)} \tag{3.7}$$

denote the corresponding invariant decomposition of the Hilbert space. In the case that $\Sigma = S^2$ and $K = T$ is a torus, this is equivalent to the decomposition (2.49).

The representations $L^2_{(n)}$ are in general far from irreducible. For example if Σ has a boundary, then the boundary will be invariant under diffeomorphisms, and one can define invariant vectors in terms of holonomy about the boundary components. Also diffeomorphisms respect the decoupling of "internal degrees of freedom"; for example in the case $\Sigma = S^2$ and $\mathfrak{k}$ abelian, we computed that as a representation

$$L^2_{(1)} = dim(\mathfrak{k})(L^2(\Sigma, \omega) \ominus \mathbb{C}1). \tag{3.8}$$

If the surface Σ is closed, and $\mathfrak{k}$ is simple, then it is tempting to conjecture that the unitary representation

$$SDiff(\Sigma) \times L^2_{(n)} \to L^2_{(n)} \tag{3.9}$$

is irreducible, but proving something like this will apparently require completely novel techniques (see [Ch] in the abelian case).

3.2. A conjectural characterization of $\{\pi_* \nu^{\hbar\omega}_{YM}\}$

Given a reduction $P_1 \subset P$, there is a natural map

$$\mathcal{G}luons(P_1) \to \mathcal{G}luons(P). \tag{3.10}$$

This is a source of invariant measures for $SDiff(\Sigma)$ acting on $\mathcal{G}luons(P)$.

Conjecture 3.1. *Suppose that Σ is closed. Then for each $\hbar \geq 0$, the invariant measure $\pi_* \nu^{\hbar\omega}_{YM}$ is ergodic with respect to the action* (3.2) *of $SDiff(\Sigma)$. Conversely, suppose that μ is a finite $SDiff(\Sigma)$-invariant measure on $\mathcal{G}luons$(for some class $C^{-\alpha}$). Then*

$$\mu = constant * \pi_* \nu^{\hbar\omega}_{YM}$$

for some $\hbar$, and possibly for some reduction of P.

A special case of this is the characterization of invariant measures on the moduli space of classical solutions, with respect to the mapping class group. The ergodicity of the canonical volume element on the moduli space with respect to the mapping class group has been checked by Goldman. The basic idea is that the moduli space (minus a set of measure zero) is a completely integrable system in a way which is equivariant with respect to a large subgroup of the mapping class group. It therefore suffices to prove that this subgroup acts ergodically on the fibres of the system, which are generically tori. This is relatively straightforward using the Fourier transform. In the case of $SU(2)$, the completely integrable system is described in detail in [JW].

4. Line bundles on C and explicit evaluation of determinants

In Section 5 we will take up the question of how to rigorously construct the unitary representations of $SDiff(\Sigma)$ which we heuristically described in Section 0.2. In this section we will mainly address geometric preliminaries. For the purposes of this paper, our main goal in this section is to obtain an analytically tractable formula for the zeta function determinant which appears as a density in (0.5). This formula is due basically to Polyakov and Wiegmann [PW] and Quillen [Q].

In Sections 4 and 5, we will assume that Σ is closed and oriented, and K is simply connected and has a simple Lie algebra. As a consequence any K-bundle on Σ is trivial. We will also normalize the AdK-invariant inner product so that the invariant form on K,

$$(1/48\pi^2)\langle\theta_{\text{MC}} \wedge [\theta_{\text{MC}} \wedge \theta_{\text{MC}}]\rangle \tag{4.1}$$

is a generator for $H^3(K, \mathbb{Z})$, where θ_{MC} is the Maurer–Cartan form. By slight abuse of notation, we will also interpret (4.1) as a left-invariant holomorphic form on G, where $\langle\cdot,\cdot\rangle$ is extended complex bilinearly to $\mathfrak{g}$. The Killing form of $\mathfrak{g}$ is the negative of the dual Coxeter number times the complex bilinear extension of $\langle\cdot,\cdot\rangle$.

4.1. *Line bundles on $\mathcal{C}_{C^0}$; theoretical description*

In this subsection we assume only that Σ is closed and oriented. Given our assumptions on K, there is a single generator for $H^2(\mathcal{C}_{C^0}, \mathbb{Z})$, where this must be interpreted as $\mathcal{K}_{C^1}$-equivariant cohomology, since $\mathcal{C}_{C^0}$ is not a manifold (see Section 2 of [AB]). The goal of this subsection is to recall Mickelsson's theoretical description of the corresponding line bundle $\mathcal{L}$.

The line bundle $\mathcal{L}$ can be realized as a quotient of the trivial line bundle

$$\mathcal{A}_{C^0} \times \mathbb{C} \to \mathcal{A}_{C^0} \tag{4.2}$$

by an explicit action of the gauge group $\mathcal{K}_{C^1}$. Following Mickelsson [Mi], we define a 1-cocycle

$$\Theta : \mathcal{K}_{C^1} \times \mathcal{A}_{C^0} \to \mathbb{R}/\mathbb{Z} \tag{4.3}$$

by

$$\Theta(g, A) = \frac{1}{8\pi^2}\left\{\int_\Sigma \langle A \wedge g^{-1}\,\mathrm{d}g\rangle + WZW(g)\right\}, \tag{4.4}$$

$$WZW(g) = \frac{-1}{6}\int_B \langle\tilde{g}^{-1}\,\mathrm{d}\tilde{g} \wedge [\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g}]\rangle, \tag{4.5}$$

where B is any oriented 3-manifold with $\partial B = \Sigma$,

$$\partial\tilde{g} = g, \quad \text{and} \quad \tilde{g} \in Map_{C^0\cap W^{3/2}}(B, K) \tag{4.6}$$

(the integral in (4.5) makes sense because $\tilde{g} \in W^{3/2}$). To see that the extension $\tilde{g}$ of $g \in \mathcal{K}_{C^1}$ exists, it suffices to check that the boundary map of Banach Lie groups

$$\partial : Map_{C^0\cap W^{3/2}}(B, K) \to Map_{C^0\cap W^1}(\Sigma, K) \tag{4.7}$$

is surjective, because $C^1 \subset C^0 \cap W^1$. Since the target group in (4.7) is connected, it suffices to check that the differential is surjective, and this follows from standard Sobolev space theory (see also Remark 4.1(4)).

It is straightforward to check that Θ defined by (4.4) is a continuous cocycle, i.e.

$$\Theta(gh, A) = \Theta(g, hAh^{-1} - (\mathrm{d}h)h^{-1}) + \Theta(h, A). \tag{4.8}$$

(see Remark 4.1(2)). Thus $\mathcal{K}$ acts equivariantly on the trivial bundle $\mathcal{A}_{C^0} \times \mathbb{C} \to \mathcal{A}_{C^0}$ by

$$g, (A, z) \to (gAg^{-1} - (\mathrm{d}g)g^{-1}, \exp(2\pi \mathrm{i}\Theta(g, A))z); \tag{4.9}$$

sections of the quotient bundle $\mathcal{L} \to \mathcal{C}_{C^0}$ are functions $\sigma : \mathcal{A}_{C^0} \to \mathbb{C}$ satisfying

$$\sigma(gAg^{-1} - (\mathrm{d}g)g^{-1}) = \exp(2\pi \mathrm{i}\Theta(g, A))\sigma(A). \tag{4.10}$$

Remarks 4.1.

(1) It appears that the natural domain of the Mickelsson cocycle is $\mathcal{K}_{W^1} \times \mathcal{A}_{W^0}$, but establishing this would require far more effort. The main technical problem is to show that the boundary map (4.7) remains surjective in the absence of continuity.

(2) For maps g which are sufficiently smooth, the cocycle (4.4) can also be written in the form

$$\Theta(g, A) = CS(\tilde{A}) - CS(\tilde{g}\tilde{A}\tilde{g}^{-1} - (\mathrm{d}\tilde{g})\tilde{g}^{-1}), \tag{4.11}$$

where $\tilde{g}$ and $\tilde{A}$ are extensions of g and A to a bounding 3-manifold B as above, and CS denotes the Chern–Simons functional

$$CS(A) = \frac{1}{8\pi^2} \int_B (\langle A \wedge F\rangle - \tfrac{1}{6}\langle A \wedge [A \wedge A]\rangle) \tag{4.12}$$

(essentially as in [RSW]). This is the most transparent way to check that (4.4) is a cocycle. But from this expression, which involves curvature, it is not as clear that Θ is defined for continuous connections.

(3) This construction is natural, in the sense that there is an explicit action

$$Diff(\Sigma) \times (\mathcal{L} \to \mathcal{C}_{W^0}) : \phi, [A, z] \to [\phi_* A, z]. \tag{4.13}$$

This is well defined because

$$\Theta(\phi_* g, \phi_* A) = \Theta(g, A) \tag{4.14}$$

for any $\phi \in Aut(P)$. It is this natural unitary action which suggests that there should be a unitary representation of $SDiff(\Sigma)$ on the L^2-sections of this bundle, as described in the introduction.

(4) The WZW term can be defined via a local expression, and for complex gauge transformations. To explain this, we first recall that if Ω is a form on a manifold, and ϕ a one-parameter family of diffeomorphisms, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\phi^* \Omega\} = \mathrm{d}\{\phi^*(\mathrm{i}_{\dot\phi}\Omega)\} + \phi^*\{\mathrm{i}_{\dot\phi}\, \mathrm{d}\Omega\}. \tag{4.15}$$

If $g = \mathrm{e}^{t\xi} \in Map_{W^1 \cap C^0}(\Sigma, G)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}WZW(g) = -\frac{1}{6}\int_B \frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}^*\langle\theta_{\mathrm{MC}} \wedge [\theta_{\mathrm{MC}} \wedge _{\mathrm{MC}}]\rangle$$
$$= -\frac{1}{2}\int_\Sigma g^*\langle\theta_{\mathrm{MC}}(\dot{g}), [\theta_{\mathrm{MC}} \wedge \theta_{\mathrm{MC}}]\rangle = \int_\Sigma \langle\xi, g^*\mathrm{d}\theta\rangle$$
$$= \int_\Sigma \langle \mathrm{d}\xi \wedge g^{-1}\mathrm{d}g\rangle = \int \left\langle \mathrm{d}\xi \wedge \left(\frac{1-\exp(-ta\,\mathrm{d}\xi)}{ta\,\mathrm{d}\xi}\right)(t\mathrm{d}\xi)\right\rangle, \tag{4.16}$$

where the second equality uses (4.15), the third uses the Maurer–Cartan equation, and the fifth uses the standard expression for the differential of the exponential map in terms of the Todd series. Upon integrating it follows that

$$WZW(\mathrm{e}^\xi) = \int \langle \mathrm{d}\xi \wedge F(a\,\mathrm{d}\xi)(\mathrm{d}\xi)\rangle, \tag{4.17}$$

where $F(\lambda) = \lambda - \sinh(\lambda)/\lambda^2$; in particular $WZW(\mathrm{e}^\xi)$ is an entire function of $\xi \in Map_{W^1 \cap C^0}(\Sigma, \mathfrak{g})$.

The main shortcoming of (4.17) is that it fails to indicate why the WZW term is well-defined mod $\mathbb{Z}$ as a function of g.

4.2. *Quillen's analytic realization of $\mathcal{L}$*

To write down an explicit section, we need to find a more concrete model for the hermitian line bundle $\mathcal{L} \to \mathcal{C}_{C^0}$. This is done using Quillen's determinant line construction [Q], which depends upon the choice of a spin structure. There are two main points to be made in this subsection and the next. The first is that Quillen's determinant line can be explicitly identified with $\mathcal{L}$ (so that in particular Quillen's construction can be extended to continuous potentials), and the other is that determinants in two dimensions can be computed explicitly, at least in gauge directions.

In this subsection we fix a Riemannian spin structure for Σ. We also fix a nontrivial holomorphic representation $G \to GL(V)$, and a unitary structure on V such that $K \to U(V)$. We then have $\langle x, y\rangle = -m\,tr_V(xy)$, for $x, y \in \mathfrak{g}$, for some positive integer m.

There is a linear isomorphism

$$\begin{aligned}&\mathcal{A}_{C^0} = \Omega^1_{C^0}(\Sigma, \mathfrak{k}) \to \mathcal{A}^{0,1}_{C^0} = \Omega^{0,1}_{C^0}(\Sigma, \mathfrak{g}) : A \to a = (A^{\mathbb{C}})^{0,1},\\ &A = a - a^*.\end{aligned} \tag{4.18}$$

Via this isomorphism, the gauge action by $\mathcal{K}$ on unitary connections extends to a holomorphic gauge action by $\mathcal{G}$, the complex gauge group, on (0,1) forms,

$$g \cdot a = gag^{-1} - (\bar{\partial}g)g^{-1}. \tag{4.19}$$

Now suppose that $a \in \mathcal{A}^{0,1}_{C^0}$. The coupled operator $\bar{\partial}_a = \bar{\partial} + a$, acting on sections of $\kappa^{1/2} \otimes V$ (where $\kappa^{1/2}$ is the holomorphic square root of the canonical bundle determined by the spin structure), can be interpreted as a Fredholm operator, and it has index zero. For $\bar\partial$ defines an index zero Fredholm operator of Hilbert spaces

$$\bar\partial : \Omega^0_{W^1}(\kappa^{1/2} \otimes V) \to \Omega^{0,1}_{W^0}(\kappa^{1/2} \otimes V), \tag{4.20}$$

and a defines a compact operator between these spaces, because the inclusion of W^1 into W^0 is compact, and a is bounded as an operator on W^0-spaces. Hence $\bar\partial + a$ is a compact perturbation of $\bar\partial$, viewed as a Fredholm operator.

The pullback of the determinant bundle on Fredholm operators gives a concrete realization of a power of the line bundle $\mathcal{L}$. Note that via the identification (4.18), $\mathcal{A}_{C^0}$ inherits a complex structure, and $\mathcal{L}$ is holomorphic. We will denote the pullback of the canonical section by $det\ \bar\partial$.

Our next task is to check that the Quillen metric extends to the determinant line over continuous potentials. The obvious approach would be to directly extend Quillen's method to continuous potentials. We have not succeeded in determining whether this direct approach is feasible, but we would like to prove a partial result in this direction.

Suppose that a is a continuous potential as above, and let

$$H_a = (\bar\partial + a)^*(\bar\partial + a). \tag{4.21}$$

This operator can be interpreted as a self-adjoint operator with domain

$$\mathcal{D}(H_a) = \{v \in \Omega^0_{W^1} = \mathcal{D}(\bar\partial + a) : (\bar\partial + a)v \in \Omega^{0,1}_{W^1} = \mathcal{D}((\bar\partial + a)^*)\}. \tag{4.22}$$

To verify this, and to effectively work with this operator, it is convenient to introduce the associated quadratic form

$$Q_a : \Omega^0_{W^1} \to \mathbb{R} : v \to |(\bar\partial + a)v|^2_{\Omega^{0,1}_{W^0}} \tag{4.23}$$

(note that the domain of Q_a does not depend upon a). To prove that $\bar\partial + a$ with domain $\Omega^0_{W^1}$ is closed as an unbounded operator from $\Omega^0_{W^0}$ to $\Omega^{0,1}_{W^0}$, and hence that H_a is self-adjoint, we must verify that $\Omega^0_{W^1}$ is a Hilbert space with respect to the norm

$$\left(Q_a(v) + |v|^2_{\Omega^0_{W^0}}\right)^{1/2}, \tag{4.24}$$

i.e. we must check that this norm is equivalent to the W^1-norm, the norm (4.24) with $a = 0$ (see Proposition 2.2 and Theorem 2.3 of [Faris]). This is easy, using the assumption that a is continuous (hence bounded):

$$Q_a(v) + |v|^2 = Q_0(v) + 2\mathbb{R}e\langle\bar\partial v, av\rangle + |av|^2 \le 2(Q_0(v) + |av|^2), \tag{4.25}$$

and similarly

$$Q_0(v) + |v|^2 \le 2(Q_a(v) + |av|^2). \tag{4.26}$$

Proposition 4.2. *Suppose that* $a \in \mathcal{A}^{0,1}_{C^0}$.

(1) *The operator* H_a *has discrete spectrum, and there is a nonzero constant* c, *independent of* a, *such that*

$$\lim(\lambda_j/j) = c,$$

where the eigenvalues $\lambda_1(H_a) \le \lambda_2(H_a) \le \cdots$ *are counted according to multiplicity.*

(2) *The function* $h_a(t) = trace(\mathrm{e}^{-tH_a})$ *has the property*

$$|h_a(t) - \frac{1}{8\pi t}(c_{-1} + c_0 t)| = \mathrm{O}(t^{-1/2}) \quad as\ t \downarrow 0,$$

where $c_{-1} = rk(V) * Area(\Sigma)$, *and* $c_0 = \frac{1}{12} rk(V) * genus(\Sigma)$.

Proof. The resolvent $(H_a - \lambda)^{-1}$ is a compact operator on $\Omega^0_{W^0}$ whenever $\lambda \notin spec(H_a)$. For the resolvent defines a bounded operator

$$\Omega^0_{W^0} \to \mathcal{D}(H_a), \tag{4.27}$$

and $\mathcal{D}(H_a)$ is compactly embedded in W^0. In particular the spectrum of H_a is discrete.

Now consider the spectral function of H_a,

$$\begin{aligned} N_a(\lambda^2) &= \#\{\lambda_j : \lambda_j \le \lambda^2\} \\ &= \sup\{dim(L) : L \subset \Omega^0_{W^1}, Q_a(v) \le \lambda^2 |v|^2_{W^0}, v \in L\}. \end{aligned} \tag{4.28}$$

If we are given a subspace L such that $Q_0(v) \le \lambda^2 |v|^2$ on L, then

$$|(\bar\partial + a)v|^2 = |\bar\partial v|^2 + 2\mathbb{R}e\langle \bar\partial v, av\rangle + |av|^2 \le (\lambda^2 + 2\lambda|a|_\infty + |a|^2_\infty)|v|^2;$$

hence $Q_a(v) \le (\lambda + |a|_\infty)^2 |v|^2$ on L. This implies that

$$N_0(\lambda^2) \le N_a((\lambda + |a|_\infty)^2), \tag{4.29}$$

and similarly

$$N_a(\lambda^2) \le N_0((\lambda + |a|_\infty)^2). \tag{4.30}$$

In particular it follows that

$$\lim \frac{N_a(\lambda)}{\lambda} = \lim \frac{N_0(\lambda)}{\lambda} \tag{4.31}$$

as $\lambda \to \infty$ (the latter limit exists), and this is equivalent to Proposition 4.2(1) with $c^{-1} = lim(N_a(\lambda)/\lambda)$ (see Proposition 13.1 and Theorem 15.2 of [Sh]).

For part (2) of Proposition 4.2, we begin by noting that since $\bar\partial$ has index zero on $\kappa^{1/2}$,

$$h_0(t) = \tfrac{1}{2} rk(V) * trace(\mathrm{e}^{-t\Delta}) = (1/8\pi t)(c_{-1} + c_0 t) + \mathrm{o}(t), \tag{4.32}$$

where here Δ is the Laplacian for $\kappa^{1/2}$, and only the $\mathrm{o}(t)$-term depends upon the metric. To prove Proposition 4.2(2), it suffices to show that

$$|h_a(t) - h_0(t)| = \mathrm{O}(t^{-1/2}), \quad \text{as } t \to 0, \tag{4.33}$$

and we can choose any metric we like to prove this. In particular we can choose the metric such that $N_0(\lambda) - \lambda$ is bounded, hence that

$$N_0((\lambda + |a|_\infty)^2) - N_0(\lambda^2) \le \alpha\lambda + \beta, \tag{4.34}$$

where α and β are constants. Now

$$h_a(t) = \int_0^\infty e^{-t\lambda}\, dN_a(\lambda) = t \int_0^\infty N_a(\lambda) e^{-t\lambda}\, d\lambda. \tag{4.35}$$

Therefore using (4.30) and (4.34),

$$\begin{aligned} h_a(t) - h_0(t) &\le t \int_0^\infty (N_0((\sqrt{\lambda} + |a|_\infty)^2) - N_0(\lambda)) e^{-t\lambda}\, d\lambda \\ &\le t \int_0^\infty (\alpha\sqrt{\lambda} + \beta) e^{-t\mu}\, d\mu = O(t^{-1/2}), \end{aligned} \tag{4.36}$$

with a similar lower bound. This proves (2). □

Let

$$\zeta_a(z) = \sum \lambda_j^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty h_a(t) t^{z-1}\, dt. \tag{4.37}$$

By Proposition 4.2(1) it follows that ζ_a is a holomorphic function in the right half plane $\{\mathbb{R}e(z) > 1\}$. By a standard argument, using Proposition 4.2(2), ζ_a has a meromorphic extension to $\{\mathbb{R}e(z) > \frac{1}{2}\}$ with a single simple pole at $z = 1$; explicitly,

$$\zeta_a(z) = \frac{1}{\Gamma(z)} \left\{ \frac{c_{-1}}{z-1} + \frac{c_0}{z} + \int_0^1 (h_a(t) - \frac{1}{8\pi}(c_{-1}t^{-1} - c_0)) t^{z-1}\, dt + \int_1^\infty h_a(t) t^{z-1}\, dt \right\}. \tag{4.38}$$

If we could improve the error term in Proposition 4.2(2) from $O(t^{-1/2})$ to $O((\log(t^{-1}))^{-p})$, for some $p > 2$, then we could assert that $\zeta_a'(0)$, extends continuously to continuous potentials, and this would imply the extension of Quillen's metric that we alluded to earlier. Instead we will have to resort to other methods.

One approach, not so useful for our purposes, is purely abstract. In [Q] Quillen proved that the curvature of the determinant line, equipped with Quillen metric, is given by

$$\bar{\partial}\partial \log |det\, \bar{\partial}|^2 = m\bar{\partial}\partial q, \tag{4.39}$$

where

$$q(a) = \frac{\mathrm{i}}{2\pi} \int \langle a \wedge a^* \rangle = \frac{1}{4\pi} \int \langle A \wedge *A \rangle. \tag{4.40}$$

Since the two form (4.39) is defined over continuous potentials, it follows by abstract theory that there is an essentially unique gauge-invariant hermitian structure over continuous potentials such that the corresponding hermitian holomorphic connection has curvature (4.39); since the restriction to smooth potentials must agree with Quillen's metric, this implies that Quillen's metric has a continuous extension.

A more concrete approach is to explicitly identify $Det\ \bar{\partial}$ with $\mathcal{L}$, as we will now do. Let $\mathcal{T}$ denote the trivial holomorphic line bundle over $\Omega^{0,1}_{C^0}$, with nontrivial hermitian metric determined by

$$|1|^2_{\mathcal{T}} = \mathrm{e}^{-mq}. \tag{4.41}$$

As observed by Quillen, there is an essentially unique unit length holomorphic section δ of $\mathcal{T} \otimes Det\ \bar{\partial}$, and in terms of this trivialization, we can define a holomorphic function of potentials a by

$$det(\bar{\partial}_a; \bar{\partial}_0) = \frac{1 \otimes det\ \bar{\partial}}{\delta}. \tag{4.42}$$

Proposition 4.3. *Sending* $\mathrm{e}^{-mq/2} \otimes \delta \in \Omega^0(\mathcal{T}^* \otimes \mathcal{T} \otimes Det\ \bar{\partial})$ *to* $1 \in \Omega^0(\mathcal{A}_{C^0} \times \mathbb{C})$ *induces isomorphisms of hermitian line bundles*

$$\begin{array}{ccccc} Det\ \bar{\partial} \to \mathcal{A}^{0,1}_{C^0} & \to & \mathcal{A}_{C^0} \times \mathbb{C} & & \\ \downarrow & & \downarrow & & \\ Det\ \bar{\partial}/\mathcal{K} \to \mathcal{A}^{0,1}_{C^0}/\mathcal{K}_{C^1} & \to & \mathcal{L}^{\otimes m} & \to & \mathcal{C}_{C^0} \end{array}$$

In particular the canonical section $det\ \bar{\partial}$ *is mapped to the function*

$$\sigma : \mathcal{A}_{C^0} \to \mathbb{C} : A \to \mathrm{e}^{-(1/2)mq(a)} det(\bar{\partial}_a; \bar{\partial}_0).$$

Proof. We need to check that the $\mathcal{K}$-invariant sections of $Det\ \bar{\partial}$ are mapped to functions that transform according to the Mickelsson cocycle. It suffices to check this for the canonical section. This follows rather directly from the calculations in [Q], as we will now explain.

Suppose that $a = a(w)$ is a holomorphic one-parameter family of smooth potentials, where $\bar{\partial}_a$ is always invertible. In [Q] Quillen calculated that

$$\begin{aligned} \frac{\partial}{\partial w} \log det_\zeta |\bar{\partial}_a|^2 &= tr\left((G - G_0)\frac{\partial a}{\partial w}\right) \\ &= \int_\Sigma tr_V \left(J(a) \wedge \frac{\partial a}{\partial w}\right), \end{aligned} \tag{4.43}$$

where

$$J(a)|_{z'} = \lim_{z \to z'} (G(z, z') - G_0(z, z')) \in \Omega^{1,0}(End(V)). \tag{4.44}$$

Here G is the Green's function for $\bar{\partial}_a$ and G_0 is a parametrix defined explicitly near the diagonal by

$$G_0(z, z') = \frac{1}{2\pi \mathrm{i}} \mathrm{d}z' \partial_{z'} (\log(r^2(z, z'))) F(z, z'), \tag{4.45}$$

where r^2 is the geodesic distance and F is parallel translation from the fiber of $\kappa^{1/2} \otimes V$ at z' to the fiber at z.

Our first task is to determine the density $J(a)$. Since G is a Green's function and G_0 is given explicitly, it is easy to find an equation satisfied by J. That is the content of Lemma 4.4.

Let ∇ denote the unique unitary connection determined by $\bar{\partial}_a$ (where $\kappa^{1/2} \otimes V$ has the product hermitian structure). In terms of a local coordinate z for Σ and a holomorphic frame for $\kappa^{1/2} \otimes V$, we have

$$\mathrm{d}s = \rho |\mathrm{d}z|, \qquad \nabla = \partial + \theta\, \mathrm{d}z + \bar{\partial}, \tag{4.46}$$

and the Taylor series expansion

$$\begin{aligned} F(z, z') = {} & 1 - \theta(z')(z - z') + \tfrac{1}{2}\{\theta(z')^2 - \partial_{z'}\theta + (\theta(z') - 1)\partial_{z'}(\log \rho)\}(z - z')^2 \\ & - \tfrac{1}{2}(\partial_{\bar{z}'}\theta)|z - z'|^2 + \mathrm{o}(|z - z'|^2). \end{aligned} \tag{4.47}$$

To derive this (supposing for simplicity that $z' = 0$), note that if $z(t)$ is a geodesic, then $z(t)$ satisfies the geodesic equation

$$\ddot{z} + \partial_z (\log \rho^2)(\dot{z})^2 = 0 \tag{4.48}$$

and $F(z(t))$, parallel translation along $z(t)$, satisfies

$$(\mathrm{d}/\mathrm{d}t) F(z(t)) + \theta(z(t))\dot{z}(t) = 0. \tag{4.49}$$

This implies that

$$\begin{aligned} F(z(t)) &= 1 - \theta(0)\dot{z}(0)t + \tfrac{1}{2}(\theta(0)^2 \dot{z}(0)^2 - \{\mathrm{d}\theta(\dot{z}(0))\dot{z}(0) + \theta(0)\ddot{z}(0)\})t^2 + \mathrm{o}(t^2) \\ &= 1 - \theta(0)\dot{z}(0)t + \tfrac{1}{2}\Big((\theta(0)^2 - (\partial_z\theta) - \partial_z \log \rho^2)\dot{z}(0)^2 \\ &\qquad - (\partial_{\bar{z}}\theta)|\dot{z}(0)|^2\Big) t^2 + \mathrm{o}(t^2). \end{aligned} \tag{4.50}$$

It is straightforward to plug the expansion

$$z(t) = \dot{z}(0)t - \partial_z(\log \rho)\dot{z}(0)^2 t^2 + \mathrm{o}(t^2) \tag{4.51}$$

into (4.49) and check that the result agrees with (4.47). This implies the validity of (4.47).

Lemma 4.4. *Let $\Omega = \nabla^2 \in \Omega^{1,1}(End V)$, the curvature of $(\kappa^{1/2} \otimes V, \nabla)$. Then*

$$\partial_{\bar{z}} J(a) + [a \wedge J(a)] = \frac{\mathrm{i}}{2\pi}\Omega.$$

Proof. In terms of the local parameter z and the holomorphic frame above, the equation $\bar\partial_a \circ G = 1$ and the transpose of the equation $G \circ \bar\partial_a = 1$ yield the local formulas

$$\partial_{\bar z} G(z, z') = \delta(z - z'), \qquad -\partial_{\bar z'} G(z, z') = \delta(z - z'), \tag{4.52}$$

respectively. Here we are using G to denote both the inverse of $\bar\partial_a$ and its local representation in our holomorphic frame; we will do the same with G_0. Using (4.47) we also have

$$\begin{aligned}\partial_{\bar z} G_0(z, z') &= \delta(z - z') + \frac{\mathrm{i}}{2\pi}\frac{1}{z - z'}\partial_{\bar z} F(z, z')\\ &= \delta(z - z') + \frac{\mathrm{i}}{2\pi}(-\tfrac{1}{2}\partial_{\bar z'}\theta) + \mathrm{o}(|z - z'|).\end{aligned} \tag{4.53}$$

Similarly,

$$\partial_{\bar z'} G_0(z, z') = -\delta(z - z') + (\mathrm{i}/2\pi)(\tfrac{1}{2}\partial_{\bar z}\theta) + \mathrm{o}(|z - z'|). \tag{4.54}$$

It follows from these calculations that

$$(\partial_{\bar z} + \partial_{\bar z'})(G - G_0)(z, z') = (\mathrm{i}/4\pi)\{\partial_{\bar z'}\theta + \partial_{\bar z}\theta\} + \mathrm{o}(|z - z'|). \tag{4.55}$$

When we take the limit $z - z' \to 0$, we obtain Lemma 4.4. □

We now complete the proof of Proposition 4.3.

Given $\xi \in \Omega^0(\Sigma, \mathfrak{k})$, let $k = \mathrm{e}^{\epsilon\xi} \in \mathcal{K}$ and $g = \mathrm{e}^{w\xi} \in \mathcal{G}$, where ϵ (resp. w) is a real (resp. complex) parameter. Assume that $\bar\partial_a$ is invertible. Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \log(\mathrm{e}^{-mq(k\cdot a)/2}\, det(\bar\partial_{k\cdot a}, \bar\partial_0))\\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\left(-\frac{im}{4\pi}\int\langle k\cdot a \wedge (k\cdot a)^*\rangle\right) + \frac{\partial}{\partial w}\Big|_{w=0} \log|det(\bar\partial_{g\cdot a}; \bar\partial_0)|^2\\ &= \frac{-m}{2\pi} Im\left(\int\langle\bar\partial_a(\xi)\wedge a^*\rangle\right) + \frac{\partial}{\partial w}\Big|_{w=0}(mq(g\cdot a) - \zeta'_{g\cdot a}(0))\\ &= \frac{-m}{2\pi} Im\left(\int(\langle\bar\partial\xi \wedge a^*\rangle)\right) - \frac{im}{2\pi}\int\langle\bar\partial_a\xi\wedge a^*\rangle - \int tr_V(J(a)\wedge\bar\partial_a\xi),\end{aligned} \tag{4.56}$$

where the first step uses the fact that Quillen's determinant is a holomorphic function of a, and the last step uses the fact that $\langle[a, \xi]\wedge a^*\rangle$ is real, because $\xi \in \mathfrak{k}$, and Quillen's calculation of the w variation of ζ' (4.43).

In Lemma 4.4 we calculated $\bar\partial_a J$ in terms of curvature, which determines J modulo the kernel of $\bar\partial_a$, acting on $\Omega^{1,0}(End(V))$. In the present calculation we can ignore both this ambiguity and the scalar part of J, since we are coupling this to the traceless operator $\bar\partial_a\xi$. Recalling that $tr_V = -m\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, we now see that (4.56) equals

$$\begin{aligned}&\frac{-m}{2\pi} Im\int\langle\bar\partial\xi\wedge a^*\rangle - \frac{im}{2\pi}\int\langle\bar\partial_a\xi\wedge a^*\rangle\\ &+\int\left\langle\frac{im}{2\pi}(\bar\partial_a)^{-1}(\partial a - \bar\partial a^* - [a^*\wedge a])\wedge\bar\partial_a\xi\right\rangle\end{aligned}$$

$$= \frac{-m}{2\pi} Im \int \langle \bar{\partial}\xi \wedge a^* \rangle + \frac{im}{2\pi} \int \langle \partial a \wedge \xi \rangle = \frac{im}{2\pi} \mathbb{R}e \int \langle a \wedge \partial \xi \rangle. \tag{4.57}$$

This calculation implies that σ satisfies the equivariance condition

$$\sigma(k \cdot a) = \exp(2\pi i m \Theta(k, A))\sigma(a). \tag{4.58}$$

This is so because if we fix A, then (4.57) shows that both sides of (4.58), as functions of k, have the same derivative at the identity (note that the derivative of the WZW term at the identity is zero). By dividing by $\sigma(a)$, since both sides of (4.58) are then 1-cocycles, this implies equality for all $k \in \mathcal{K}$. □

4.3. *On calculating Quillen's determinant and* $det_\zeta |\bar{\partial}|^2$

In the proof of Proposition 4.3 we calculated the effect of a unitary gauge transformation on σ. Let $g = e^{w\xi}$, where ξ is now allowed to have values in $\mathfrak{g}$. The same calculation shows that

$$\begin{aligned} \frac{\partial}{\partial w}\Big|_{w=0} \log(det(\bar{\partial}_{g \cdot a}; \partial_0)) &= \frac{\partial}{\partial w}\Big|_{w=0} (mq(g \cdot a) - \zeta'(0)) \\ &= \frac{im}{2\pi} \int \langle a \wedge \partial \xi \rangle. \end{aligned} \tag{4.59}$$

This determines Quillen's determinant function in gauge directions.

Proposition 4.5. *We have*

$$det(\bar{\partial}_{g \cdot a}; \bar{\partial}_0) = \exp(2\pi i m \Phi(g, a)) det(\bar{\partial}_a; \bar{\partial}_0)$$

for all $g \in \mathcal{G}$, *where*

$$\Phi(g, a) = \Theta(g, a) + \lambda(g, a),$$

$$\lambda(g, a) = \frac{1}{8\pi^2} \int \langle g \cdot a \wedge (\partial g) g^{-1} \rangle.$$

Proof. We first observe that

$$2\pi i m \frac{\partial}{\partial w}\Big|_{w=0} \Phi(e^{w\xi}, a) = \frac{im}{2\pi} \int \langle a \wedge \partial \xi \rangle, \tag{4.60}$$

which is the same as the logarithmic derivative in (4.59). It then remains to check that Φ is a $\mathbb{C}/\mathbb{Z}$-valued 1-cocycle for the action of $\mathcal{G}$ on $\mathcal{A}^{0,1}_{C^0}$. Since Θ is a $\mathbb{C}/\mathbb{Z}$-valued 1-cocycle for the action of $\mathcal{G}$ on complex 1-forms (i.e. (3.7) is valid for g, h and A complex), this reduces to checking that

$$\lambda(gh, a) = \lambda(g, hah^{-1} - (\bar{\partial} h)h^{-1}) + \lambda(h, a) + \frac{1}{8\pi^2} \int \langle (\partial h) h^{-1} \wedge g^{-1} \bar{\partial} g \rangle, \tag{4.61}$$

and this is straightforward. □

On the Riemann sphere this result in a sense explicitly determines Quillen's determinant function. For in the case of the sphere, a generic a will be in the $\mathcal{G}$-orbit of 0. It follows from Proposition 4.5 that

$$det(\bar{\partial}_a; \bar{\partial}_0) = \exp(2\pi \mathrm{i} S(g)), \tag{4.62}$$

$$S(g) = \Theta(g, -g^{-1} \cdot 0) = \frac{1}{8\pi^2} \left\{ - \int \langle g^{-1}\bar{\partial} g \wedge g^{-1} \partial g \rangle + WZW(g) \right\}, \tag{4.63}$$

where $a = g \cdot 0$. This is essentially the formula written down by Polyakov and Wiegmann in [PW] (which is remarkable, because the meaning of the determinant in their work was not specified). Note that using Remark 4.1(4), it is easy to write down the Taylor series expansion for S, hence the series for Quillen's determinant.

The functional S (which is closely related to the Wess–Zumino–Novikov–Witten action) satisfies the equation

$$S(gh) = S(g) + S(h) - \frac{2}{8\pi^2} \int \langle (\bar{\partial} h) h^{-1} \wedge g^{-1} (\partial g) \rangle . \tag{4.64}$$

Let $\mathcal{E}$ denote the complexification of the standard energy function

$$\mathcal{E} : Map_{W^1}(\Sigma, G) \to \mathbb{C} : g \to \frac{1}{2} \int \langle g^{-1}\, \mathrm{d}g \wedge * g^{-1}\, \mathrm{d}g \rangle . \tag{4.65}$$

Note that $\mathcal{E}$ is positive on $\mathcal{K}$ and negative on maps with values in $\exp(\mathfrak{p})$, $\mathfrak{p} = \mathrm{i}\mathfrak{k}$. Note also that WZW is unambiguously defined as a $\mathrm{i}\mathbb{R}$-valued function on $\exp(\mathfrak{p})$ by (4.17).

Proposition 4.6. *If* $a = g \cdot 0 = g\bar{\partial}(g^{-1})$, *then*

$$|\sigma(A)|^2 = \exp(4\pi m \mathrm{i} S(\Omega)) = \exp\Big((m/2\pi)(\mathcal{E}(\Omega) + \mathrm{i} WZW(\Omega))\Big),$$

where $\Omega = g^* g$.

Proof. It is easy to check that

$$\begin{aligned} &WZW(g^*) = -WZW(g)^* \bmod 8\pi^2\mathbb{Z}, \\ &S(g^*) = -S(g)^* \bmod \mathbb{Z}. \end{aligned} \tag{4.66}$$

Hence (4.6.4) implies that

$$\begin{aligned} S(g^* g) &= 2\mathrm{i}\, Im\, S(g) - \frac{2}{8\pi^2} \int \langle (\bar{\partial} g) g^{-1} \wedge g^{*-1} \partial g^* \rangle \\ &= 2\mathrm{i}\, Im\, S(g) - \frac{2}{8\pi^2} \int \langle a \wedge a^* \rangle = 2\mathrm{i}\, Im \left\{ S(g) - \frac{1}{8\pi^2} \int \langle a \wedge a^* \rangle \right\}. \end{aligned} \tag{4.67}$$

Since

$$\sigma(A) = \exp\left(-\frac{\mathrm{i}m}{4\pi} \int \langle a \wedge a^* \rangle + 2\pi \mathrm{i} S(g) \right), \tag{4.68}$$

(4.67) implies Proposition 4.6. □

Now suppose that Σ is a closed Riemann surface of positive genus. It is no longer the case that the complex gauge orbit of $a = 0$ is dense; instead one must consider all connections having curvature zero. For a continuous connection A, curvature F_A is not defined. However, meaning can be attached to the equation $F_A = 0$; it means that for the corresponding gluon (or holonomy function) h^A, the value of h^A_c depends only upon the homotopy class of c.

There is a natural injective map

$$\mathcal{G}_{C^1} \times_{\mathcal{K}_{C^1}} \{A \in \mathcal{A}_{C^0}: F_A = 0\} \to (\mathcal{A}^{0,1}_{C^0}): (g, A_0) \to A = g \cdot a_0; \tag{4.69}$$

the image of this map is all semistable potentials; one can consult [AB] for the definition of semistable in this context, but for our purposes it suffices to take the image of (4.69) as the definition. Because of the injectivity of (4.69), the set of semistable potentials has the structure of a principal fibre bundle with fiber $\mathcal{G}_{C^1}$ and base $H^1(\Sigma, K)$; the projection to the base is given by

$$(g, A_0) \to h^{A_0}. \tag{4.70}$$

In this paper we will not take up the question of how to compute Quillen's determinant in full generality; this involves finding explicit local cross-sections for the projection

$$\{A_0 \in \mathcal{A}: F_{A_0} = 0\} \to H^1(\Sigma, K), \tag{4.71}$$

and some further considerations about the density $J(a)$ of the preceding subsection. In this paper we are more interested in $|\sigma(A)|$; because the quantity $|\sigma(A_0)|$ is gauge invariant with respect to unitary transformations, we have the following generalization of Proposition 4.6.

Proposition 4.7. *If $a = g \cdot a_0$, then*

$$|\sigma(A)|^2 = \exp\Big((m/2\pi)(\mathcal{E}(\Omega) + \mathrm{i}WZW(\Omega) + \Psi(\Omega, a_0))\Big)|\sigma(A_0)|^2,$$

where

$$\Psi(\Omega, a_0) = \mathrm{i}\int \langle \Omega \cdot a_0 \wedge a_0^* - \Omega a_0 \Omega^{-1} \wedge (\Omega^{-1} \cdot a_0)^* \rangle,$$

$\Omega = g^* g$ and $|\sigma(A_0)|^2$ descends to define a smooth function on the moduli space $H^1(\Sigma, K)$.

The proof of this is a straightforward calculation along the same lines as Proposition 4.6.

5. Line bundles and quantization, probabilistic considerations

In the previous section we found relatively explicit formulas for $det_\zeta|\bar\partial|^2$ (or at least we reduced the problem of finding such formulas to finite-dimensional considerations). In this section we will discuss the problem of coupling this function to the Yang–Mills measure, i.e. making sense of the measure given by the formal expression (0.4).

5.1. Heuristics

Consider first the case of the sphere. Using Proposition 4.6 we can rewrite the heuristic expression (0.4) as

$$(1/E)\exp(\mathcal{E}(\Omega)+\mathrm{i}WZW(\Omega))\,\mathrm{d}\pi_*\nu_{YM}(A), \tag{5.1}$$

where

$$\Omega=g^*g, \qquad -(\bar\partial g)g^{-1}=a, \qquad A=a-a^*. \tag{5.2}$$

There is a simple heuristic expression for the measure $\pi_*\nu_{YM}$ in terms of the coordinate Ω, because

$$\langle F_A\wedge *F_A\rangle=\langle\bar\partial(\Omega^{-1}(\partial\Omega))\wedge *\bar\partial(\Omega^{-1}(\partial\Omega))\rangle. \tag{5.3}$$

To see this, use the relation

$$g^{-1}(a-a^*)g+g^{-1}\,\mathrm{d}g=\Omega^{-1}\partial\Omega. \tag{5.4}$$

This implies that

$$g^{-1}F_Ag=\bar\partial(\Omega^{-1}\partial\Omega), \tag{5.5}$$

which in turn implies (5.3). The relation (5.4) is easy to check directly; it also follows abstractly from the fact that $d+A$ is the coordinate expression for the unique holomorphic unitary connection in the bundle $\Sigma\times V\to\Sigma$, where the unitary structure is constant, and the holomorphic structure is gotten by declaring ϵg to be a holomorphic frame, where ϵ denotes a constant frame; in the holomorphic frame ϵg the connection is given by

$$\partial+\Omega^{-1}\partial\Omega+\bar\partial. \tag{5.6}$$

Thus on a heuristic level, in terms of the correspondence (5.2)

$$\mathrm{d}\nu_{YM}(A)=(1/E)\exp(-\tfrac{1}{2}\langle\bar\partial(\Omega^{-1}(\partial\Omega))\wedge *\bar\partial(\Omega^{-1}(\partial\Omega))\rangle)\mathcal{D}g, \tag{5.7}$$

$$\mathrm{d}\pi_*\nu_{YM}([A])=(1/E)\exp(-\tfrac{1}{2}\langle\bar\partial(\Omega^{-1}(\partial\Omega))\wedge *\bar\partial(\Omega^{-1}(\partial\Omega))\rangle)\mathcal{D}\Omega, \tag{5.8}$$

where $\mathcal{D}g$ denotes the fictitious Haar measure for $\mathcal{G}$, and $\mathcal{D}\Omega$ the invariant measure on $Map(\Sigma,\exp(\mathfrak{p}))$.

Now suppose that Σ is a closed Riemannian surface of positive genus. Recall that there is a fibration

$$\begin{array}{c} \mathcal{G}_{C^1}\to(\mathcal{A}_{C^0})_{ss}\equiv\mathcal{G}_{C^1}\times_K\{A\in\mathcal{A}_{C^0}:F_A=0\} \\ \downarrow \\ H^1(\Sigma,K) \end{array} \tag{5.9}$$

To find the corresponding disintegration of ν_{YM} (on a heuristic level), we compute the Yang–Mills functional for

$$a=g\cdot a_0, \tag{5.10}$$

where $g \in \mathcal{G}$ and A_0 has curvature zero. A (rather lengthy) calculation shows that

$$g^{-1}F_A g = g^{-1}(\partial(g \cdot a_0) - \bar{\partial}((g \cdot a_0)^*) - a \wedge a^* - a^* \wedge a)g$$
$$= F_{a_0-(\Omega^{-1}\cdot a_0)^*} \tag{5.11}$$

One therefore expects that

$$\nu_{YM} = \int\limits_{H^1(\Sigma,K)} \left\{\exp\left(-\frac{1}{2}\int \langle F_{a_0-(\Omega^{-1}\cdot a_0)^*} \wedge F_{a_0-(\Omega^{-1}\cdot a_0)^*}\rangle\right)\mathcal{D}g\right\} \rho \, \mathrm{d}P([a_0]). \tag{5.12}$$

where ρ is a density and $\mathrm{d}P$ denotes the canonical symplectic volume element on moduli space.

To have a minimal understanding of these measures, say in the case of S^2, we should be able to do all of the following. First we should be able to show that the formal expression (5.8) does indeed define a measure on $Map_{C^0}(\Sigma, \exp(\mathfrak{p}))$. Secondly, we should be able to show that a probabilistic extension of the correspondence

$$h^A \leftarrow [A] \leftarrow \Omega \tag{5.13}$$

defines an isomorphism which identifies $\pi_* \nu_{YM}$ and the measure corresponding to the formal expression (5.8). Finally we should be able to show that the formal expression (5.1) can be interpreted as a measure which is absolutely continuous with respect to the image of $\pi_* \nu_{YM}$ under the correspondence (5.13).

5.2. *The abelian analogue*

To gain a feeling for the analytical properties of the measures represented by the formal expressions (5.1) and (5.8), it is instructive to consider the case $\mathfrak{k} = \mathrm{i}\mathbb{R}$; in this case both formal expressions represent Gaussian measures. We will write down the finite-dimensional distributions, with respect to evaluation at points, and check that (5.1) is absolutely continuous with respect to (5.8).

Let $\mathrm{i}f = \log \Omega$. Then (5.8) is essentially equivalent to

$$\exp\left(-\frac{1}{2}\int\limits_{\Sigma} *|\Delta f|^2\right)\mathcal{D}f, \tag{5.14}$$

where Δ denotes the scalar Laplace–Beltrami operator. This formal expression (5.14) represents the Gaussian measure associated to the Hilbert space $W^2_{E=0}$, real-valued W^2-functions on Σ with expectation zero, where the inner product is given by

$$f \cdot g = \int\limits_{\Sigma} *(\Delta f \Delta g). \tag{5.15}$$

It is well known that this measure is supported on continuous functions with expectation zero.

Now suppose that Π denotes a configuration on Σ, i.e. a finite subset of Σ. Our task is to compute the projection of the Gaussian with respect to the evaluation map

$$eval^{\Pi} : \left\{ f \in C(\Sigma, \mathbb{R}) : \int f = 0 \right\} \to \mathbb{R}^{|\Pi|} : f \to (f(q))_{q \in \Pi}. \tag{5.16}$$

This image measure is given by

$$(1/E) \exp(-\tfrac{1}{2} C^{-1} \vec{q} \cdot \vec{q}) \prod \mathrm{d}q_j, \tag{5.17}$$

where C is the covariance matrix. Given a point $q \in \Sigma$, the evaluation function is represented by a function $\eta = \eta_q \in W^2_{E=0}$, in the sense that

$$f(q) = \int *(\Delta f \Delta \eta) \tag{5.18}$$

for all $f \in W^2_{E=0}$. We then have

$$C_{q,p} = \eta_q \cdot \eta_p = \int_{\Sigma} *(\Delta \eta_q \Delta \eta_p). \tag{5.19}$$

Eq. (5.18) says that the function $h = \Delta \eta$ satisfies the distributional equation

$$\Delta(h) = \delta_q + constant. \tag{5.20}$$

This equation and the constraint $\int *h = 0$ uniquely determine h.

If Σ is hyperbolic, then there exists a Green's function with singularity at q, G_q. We have

$$h = G_q - \int *G_q, \tag{5.21}$$

and the constant in (5.20) is zero. Hence in these cases

$$C_{q,p} = \int *(G_q G_p) - \left(\int *G_q \right) \left(\int *G_p \right). \tag{5.22}$$

Suppose that Σ is the sphere with the standard metric. In terms of the stereographic coordinate $z = r \exp(\mathrm{i}\theta)$, the metric and Laplacian are given by

$$4(1 + r^2)^{-2} |\mathrm{d}z|^2, \qquad 4\Delta_{S^2} = (1 + r^2)^2 \Delta_{\mathbb{R}^2}. \tag{5.23}$$

To solve Eq. (5.20), we take q to correspond to $z = 0$. Let $h = \Delta_{S^2} \eta$; h is a function of r alone. Eq. (5.20) is equivalent to

$$(1 + r^2)^2 (\partial_r^2 + (1/r)\partial_r) h = 4\delta_0 + constant, \tag{5.24}$$

where h must have a limit as $r \to \infty$, and $\int *h = 0$. We find that

$$h(r) = (1/4\pi)(-\ln(r^2) + \ln(1 + r^2)) - c. \tag{5.25}$$

[One easily checks that

$$(1 + r^2)^2 (\partial_{rr} + (1/r)\partial_r) \ln(1 + r^2) = 4.]$$

In intrinsic geometric terms, this means that

$$\eta_q(p) = -(1/4\pi)\ln(\sin^2 \tfrac{1}{2}d), \tag{5.26}$$

where d is the distance between p and q.

In stereographic coordinates the distance between two points is given by

$$d(z, w) = 2\arctan\left|\frac{z-w}{1+\bar{z}w}\right|. \tag{5.27}$$

Given q_1 and q_2,

$$c_{q_1,q_2} = \frac{1}{(4\pi)^2}\int\int \ln\left(\frac{1+r^2}{r^2}\right)\ln\left(\frac{R^2+r^2+r^2R^2+1}{R^2+r^2-2rR\,\cos(\theta)}\right)\frac{4r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta. \tag{5.28}$$

In the case of the standard torus $\Sigma = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, with $q = 0 \in \Sigma$, we have

$$h = \sum_{k\neq 0}\frac{1}{|k|^2}\mathrm{e}^{ik\cdot x} \tag{5.29}$$

and

$$c_{q_1,q_2} = c_{0,q_1-q_2} = \sum_{k\neq 0}\frac{1}{|k|^4}\mathrm{e}^{ik\cdot(q_1-q_2)}. \tag{5.30}$$

The coupled measure (5.1) (where we have inserted a coupling parameter) is given formally by the expression

$$(1/E)\exp\left(-\frac{1}{2}\int_{\Sigma} *\{|\Delta f|^2 + \beta|\mathrm{d}f|^2\}\right)\mathcal{D}f. \tag{5.31}$$

One can make sense of this as the Gaussian measure for the space $W^2_{E=0}$, where the inner product is now given by

$$f\cdot g = \int_{\Sigma} *(|\Delta f|^2 + \beta|\mathrm{d}f|^2). \tag{5.32}$$

Let $h = \Delta\eta_q$ as before. Then the equation that we must solve is

$$\Delta(h) + \beta h = \delta_q + constant. \tag{5.33}$$

For $\beta > 0$, there is a unique function G_q^β such that

$$(\Delta + \beta)G_q^\beta = \delta_q. \tag{5.34}$$

Therefore

$$\eta_q = \Delta^{-1}\left(G_q^\beta - \int *G_q^\beta\right) \tag{5.35}$$

(where we have restricted Δ to the orthogonal complement of the constants). Hence the covariance matrix is given by

$$C_{q,p} = \eta_q \cdot \eta_p = \int *(G_q^\beta G_p^\beta) - \left(\int *G_q^\beta\right)\left(\int *G_p^\beta\right) + \beta \int \mathrm{d}\eta_q \wedge *\mathrm{d}\eta_p. \quad (5.36)$$

We now check absolute continuity.

Fix an orthonormal basis of real eigenfunctions ϵ_n for Δ on Σ. Let λ_n denote the eigenvalues, and f_n the coefficients of f with respect to this basis. The measure represented by (5.8) is equal to

$$\prod \frac{\lambda_n}{\sqrt{4\pi}} \exp(-\tfrac{1}{2}\lambda_n^2 f_n^2)\, \mathrm{d}f_n. \quad (5.37)$$

The coupled measure (5.1) is equal to the product measure

$$\prod \frac{\sqrt{\lambda_n^2 + \beta\lambda_n}}{\sqrt{4\pi}} \exp(-\tfrac{1}{2}(\beta\lambda_n f_n^2 + \lambda_n^2 f_n^2))\, \mathrm{d}f_n. \quad (5.38)$$

Since the f_n are independent, it is straightforward to check that these measures are equivalent. In fact the Radon–Nikodym derivative of (5.38) with respect to (5.37) equals

$$\left\{\prod (1 + \frac{\beta}{\lambda_n})^{1/2} \mathrm{e}^{-\beta/2\lambda_n}\right\} \exp\left[-\beta/2 \sum \lambda_n (f_n^2 - E(f_n^2))\right], \quad (5.39)$$

where $E(\cdot)$ denotes expectation with respect to (5.37). This makes sense as a random variable with respect to (5.37) because (1) the existence of the limit $n^{-1}\lambda_n \to$ *constant* implies that the product in (5.39) converges and; (2) the random variables $\lambda_n(f_n^2 - E(f_n^2))$ are independent, have mean zero, and the sum of their L^2-norms (with respect to (5.39)) is (easily checked to be) finite.

5.3. *The Yang–Mills measure in terms of Ω*

How do we make sense of the measure (5.8)? The basic problem is to characterize the finite-dimensional distributions with respect to evaluation. This is of broad interest for the following reason. We can write

$$\bar{\partial}(\Omega^{-1}\partial\Omega) = \mathrm{d}(*(\Omega^{-1}\,\mathrm{d}\Omega)) - \tfrac{1}{2}\mathrm{i}[\Omega^{-1}\,\mathrm{d}\Omega \wedge \Omega^{-1}\,\mathrm{d}\Omega].$$

The term $*\mathrm{d}(*(\Omega^{-1}\,\mathrm{d}\Omega))$ is essentially the gradient of the energy function $\mathcal{E}$ at Ω. So if we ignore the bracket, the most important part of the measure is

$$\exp(-\tfrac{1}{2}\nabla\mathcal{E} \cdot \nabla\mathcal{E})\mathcal{D}\Omega.$$

This formal expression makes sense for $M = \exp(\mathfrak{p})$ (as a Riemannian symmetric space) replaced by a general Riemannian manifold M. The solvability of Yang–Mills suggests that it may be possible to characterize the distributions of this more general object.

Appendix A. The space of coordinate based gluons for S^3

In dimension 3 the Yang–Mills functional involves the entire metric, and in dimension 4 it depends upon the conformal class of the metric. For the invariant metrics on the 3 and 4 spheres, it seems plausible that the parametrization of based gluons using paths in spherical coordinates will be useful in understanding the Feynmann measure. Since the arguments in Section 1.3 extend directly to these cases, we will state the results.

The 3-Sphere. We consider spherical coordinates for the 3-sphere. This is the parameterization

$$(\phi_1, \phi_2, \phi_3) \to (\cos(\phi_1), \sin(\phi_1)(\cos(\phi_2), \sin(\phi_2)(\cos(\phi_3), \sin(\phi_3)))), \tag{A.1}$$

where $0 < \phi_1 < \pi, 0 < \phi_2 < \pi, 0 < \phi_3 < 2\pi$. The metric is given by

$$(\mathrm{d}s)^2 = (\mathrm{d}\phi_1)^2 + \sin^2(\phi_1)\{(\mathrm{d}\phi_2)^2 + \sin^2(\phi_2)(\mathrm{d}\phi_3)^2\}. \tag{A.2}$$

Now suppose that g is a gluon potential for the trivial bundle $S^3 \times K$. Define

$$\begin{aligned} g^1(\phi_1, \phi_2, \phi_3) &= g_{\{t \to (t\phi_1, \phi_2, \phi_3)\}}, \\ g^2(\phi_1, \phi_2, \phi_3) &= g_{\{t \to (\phi_1, t\phi_2, \phi_3)\}}, \\ g^3(\phi_1, \phi_2, \phi_3) &= g_{\{t \to (\phi_1, \phi_2, t\phi_3)\}}. \end{aligned} \tag{A.3}$$

Proposition A.1. *The map*

$$g \to (g^1, g^2, g^3)$$

defines an isomorphism

$$\begin{aligned} \mathcal{G}luons^{(coord)} \to D_0^3 K &\times Path_{\phi_1}^{1,1}(Path_{\phi_2}^{1,*}(Path_{\phi_3}^{*=*}K)) \\ &\times Path_{\phi_1}^{1,1}(Path_{\phi_2}^{1,1}(Path_{\phi_3}^{1,*}K)). \end{aligned}$$

To parameterize the space of coordinate based gluons, given a gluon potential g, define

$$\begin{aligned} \gamma(\phi_2, \phi_3) &= g^1(1, \phi_2, \phi_3)^{-1} \circ g^1(1, 0, 0), \\ h^{\psi}(\phi_1, \phi_2, \phi_3) &= g^1(\phi_1, \phi_2, \phi_3)^{-1} \circ g^{\psi}(\phi_1, \phi_2, \phi_3) \circ g^1(\phi_1, 0, 0), \end{aligned} \tag{A.4}$$

where $\psi = 2, 3$.

Proposition A.2. *The map*

$$[g] \to h = (h^2, h^3)$$

defines an isomorphism

$$\mathcal{G}luons_{based}^{(coord)} \to \Big\{ h \in Path_{\phi_1}^{1,*} Path_{\phi_2}^{1,*} Path_{\phi_3}^{*=*} K \times Path_{\phi_1}^{1,*} Path_{\phi_2}^{1,1} Path_{\phi_3}^{1,*} K : h^2|_{\phi_1=\pi} = h^3|_{\phi_1=\pi} = \gamma \in P\text{-}component \subset Map(S^2, K)/K \Big\}$$

This is a fibre bundle with contractible fibre and base $(\Omega^2 K)_P$.

The proof is exactly the same as for Proposition 1.7. The extension to higher-dimensional spheres is straightforward.

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